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Admissible groups and cluster structures of families of log Calabi-Yau surfaces

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Admissible groups and cluster structures of families of log Calabi-Yau surfaces

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Dedication

For Almighty God whose eternal love is my true home and for mama, my only tie to this fleeting world

In the air, there your root remains,

there, in the air

- Paul Celan

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Abstract

Admissible groups and cluster structures of families of log Calabi-Yau surfaces

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We describe the action of the admissible groups on the scattering diagrams for cluster varieties associated to positive log Calabi-Yau surfaces. By introducing the technique of folding to scattering diagrams, we give a functorial construction of locally closed sub-cluster varieties. In cases of universal families of positive log Calabi-Yau surfaces, the folding procedure allows us to study the full theta bases and scattering diagrams of degenerate subfamilies, overcoming the technical difficulty of infinite wall-crossings that appear when we restrict to degenerate loci. As a byproduct, we construct examples of cluster varieties with non-equivalent cluster structures.

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CHAPTER 1

Introduction

The central theme we explore in this thesis is that disjoint subfans in cluster scattering diagrams parametrize non-equivalent cluster structures on log Calabi-Yau varieties. The main examples we investigate are cluster varieties that are families of log Calabi-Yau surfaces. The technical foundation *folding* is built up in Chapter 2 of this thesis and the main results are summarized in section 1.1. In Chapter 3 of the thesis, we apply the folding technique as developed in Chapter 2 to families of log Calabi-Yau surfaces and their degenerate subfamilies, generalizing the key example we investigate in Chapter 2. Main results in Chapter 3 are summarized in section 1.2.

1.1. Cluster structures and subfans in the scattering diagrams

In the 90's, Lusztig did seminal work on total positivity in reductive groups and canonical bases in quantum groups ([**Lu90**, **Lu94**]). *Cluster algebras* ([**FZ02**]) were invented by Fomin and Zelevinsky in 2000 as an algebraic and combinatorial framework to study total positivity and Lusztig's *dual canonical bases* in semisimple Lie groups. It turns out that cluster structures are phenomenal - they appear in many apparently unrelated areas in mathematics like Teichmüller theory, Poisson geometry, quiver representations, quantum field theory, discrete dynamical systems, combinatorics.

In 2014, Gross, Hacking, Keel and Kontsevich in [**GHKK**] proposed a new framework to understand cluster algebras using machinery from *mirror symmetry*. Mirror symmetry originated from string theory and reveals surprising duality between complex geometry and symplectic geometry on pairs of *Calabi-Yau varieties* that are mirror dual to each other. Calabi-Yau varieties are complex manifolds with a holomorphic volume form that is unique up to scaling. *Cluster varieties*, varieties obtained by gluing algebraic tori together via a special class of birational maps called *mutations* which are encoded by cluster combinatorics, are examples of open Calabi-Yau varieties called *log Calabi-Yau varieties*. Using combinatorial gadgets from enumerative geometry like *scattering diagrams* and *broken lines* - tropical analogues of holomorphic discs in symplectic geometry, Gross, Hacking, Keel and Kontsevich construct *canonical bases* for cluster algebras consisting of elements called *theta functions*. The mirror symmetry program for log Calabi-Yau varieties strongly suggest that canonical bases are intrinsic to the geometry of cluster varieties. So natural questions arise:

(1) Does there exist a log Calabi-Yau variety with multiple non-equivalent cluster structures?
(2) If so, does the theta basis depend on the choice of the cluster structure?

In this thesis, we mainly try to address the first question. The main technical tools we use are scattering diagrams as introduced in [**GHKK**]. Given a cluster variety V, let $V^\vee$ be its *Fock-Goncharov mirror dual variety*. Let $V^\vee(\mathbb{R}^T)$ be the tropicalization of $V^\vee$, an integral affine manifold of the same real dimension as $V^\vee$. One of the major contributions of [**?**] is to introduce a wall-crossing structure called a *scattering diagram* on $V^\vee(\mathbb{R}^T)$. The scattering diagram on $V^\vee(\mathbb{R}^T)$ has a region called the *cluster complex* Δ^+ that has a simplicial fan structure which is uncommon in general scattering diagrams. The integer tropical points in Δ^+ parametrize a subset of regular functions on V called *cluster variables* and yield a cluster structure on V. If a scattering diagram has multiple disjoint subfan structures, we might hope that these disjoint subfan structures can yield distinct cluster structures. The first example that I have investigated is a well-known example in cluster theory and Teichmüller theory: cluster varieties that parametrize PGL_2-local systems on once-punctured torus $\mathbb{T}_1^2$. Let $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ be the cluster $\mathcal{A}_{\mathrm{prin}}$-variety associated to $\mathbb{T}_1^2$. It is known that in this case the scattering diagram for $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ has two disjoint subfans, one is the cluster complex Δ^+ consisting of *positive chambers*, the other the complex Δ^- of *negative chambers*. Since Δ^- is not part of the cluster complex, if integer tropical points in Δ^- also parametrize regular functions on the cluster variety $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$, it will be natural to expect that Δ^- will yield a cluster structure distinct from that of Δ^+. However, any path from Δ^+ to Δ^- involves infinite wall-crossings. It is not clear whether theta functions parametrized by integer tropical points in Δ^- are formal sums or regular functions of $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$. To overcome this technical difficulty, in my thesis, I introduce techniques of folding in cluster theory to scattering diagrams. Given a seed $\mathbf{s}$, if a seed $\overline{\mathbf{s}}$ can obtained from $\mathbf{s}$ through a procedure called *folding*, then we prove the following relationship between the corresponding scattering diagrams:

THEOREM 1.1.1. *(Theorem 2.2.19 and Corollary 2.2.20) The scattering diagram $\mathfrak{D}_{\overline{\mathbf{s}}}$ can be obtained from $\mathfrak{D}_{\mathbf{s}}$ via a quotient construction.*

An ideal triangulation of the 4-punctured sphere $\mathbb{S}_4^2$ yields a seed that can be folded to a seed obtained from an ideal triangulation of $\mathbb{T}_1^2$. The application of the folding procedure to the case of cluster varieties associated to $\mathbb{T}_1^2$ yields the following theorems:

THEOREM 1.1.2. *(Theorem 2.5.7) If we build a variety $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ using torus charts parametrized by all chambers in $\Delta^+ \bigcup \Delta^-$, then $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ has two non-equivalent cluster structures corresponding to Δ^+ and Δ^- respectively.*

Thus, we already give an positive answer to the first question.

1.2. Admissible groups and cluster structures of families of log Calabi-Yau surfaces

In Chapter 3 of the thesis, we apply the folding technique as developed in Chapter 2 to families of log Calabi-Yau surfaces and their degenerate subfamilies. A *Looijenga pair* (Y, D) is a smooth projective surface Y with $D \in |-K_Y|$ a cycle of rational curves, that is, $D = D_1 + D_2 + \cdots + D_n$ where each D_i is a rational curve. Let

$$D^{\perp} := \{\alpha \in \operatorname{Pic}(Y) \mid \alpha \cdot [D_i] = 0 \text{ for all } i\}$$

Let

$$T_{(D^{\perp})^*} := \left(D^{\perp}\right)^* \otimes \mathbb{C}^*$$

We say (Y, D) is *positive* if the intersection matrix given by $(D_i \cdot D_j)$ is *not* negative semidefinite. We say that a positive Looijenga pair (Y, D) is *generic* if none of the roots in $D^{\perp}$ can be represented by an effective curve class. Given a positive generic Looijenga pair (Y, D), there is a corresponding cluster variety of $\mathcal{X}$-type fibered over $T_{(D^{\perp})^*}$ such that the generic fiber is a log Calabi-Yau surface deformation equivalent to $U := Y \backslash D$ (see section 5 of [**GHK15**] for more details). We denote this $\mathcal{X}$-cluster variety by $\mathcal{X}_{D^{\perp}}$. We can view $\mathcal{X}_{D^{\perp}}$ as the universal family of U. Since the theory of cluster varieties that are acyclic or of finite type is well-understood, we will focus on the cases where $\mathcal{X}_{D^{\perp}}$ is *not* acyclic or of finite type, or equivalently, where $D^{\perp} \simeq \mathbf{D}_n (n \geqslant 4)$ or $\mathbf{E}_n (n = 6, 7, 8)$. Let $\mathfrak{D}_{D^{\perp}}$ be the scattering diagram that lives on $\mathcal{X}_{D^{\perp}}(\mathbb{R}^T)$. A natural question arises: Does the action of the Weyl group of $D^{\perp}$ on $\mathcal{X}_{D^{\perp}}$ preserve the cluster complex in $\mathfrak{D}_{D^{\perp}}$? If not, then we may hope that the action of the Weyl group on $\mathfrak{D}_{D^{\perp}}$ will give us subfans disjoint from the cluster complex. It turned out that our hope is false, the Weyl group preserves the cluster complex:

THEOREM 1.2.1. *(Theorem 3.4.2) Let (Y, D) be a generic Looijenga pair with $D^\perp \simeq \mathbf{D}_n (n \geqslant 4)$ or $\mathbf{E}_n (n = 6, 7, 8)$. Denote by $\mathcal{X}_{D^\perp}$ the corresponding $\mathcal{X}$-cluster variety and $\mathfrak{D}_{D^\perp}$ the scattering diagram living on $\mathcal{X}_{D^\perp}(\mathbb{R}^T)$. Then the admissible group of (Y, D) acts faithfully on the scattering diagram $\mathfrak{D}_{D^\perp}$ and preserves the cluster complex.*

However, the folding technique we developed in Chapter 2 applies immediately to $\mathcal{X}_{D^\perp}$ when $D^\perp \simeq \mathbf{D}_n (n \geqslant 4)$. Let $\mathbf{s}$ be the seed we use to construct $\mathcal{X}_{\mathbf{D}_n}$ and $\bar{\mathbf{s}}$ the folded seed. The $\mathcal{X}$-cluster variety corresponding to $\bar{\mathbf{s}}$ is a locally closed subvariety of $\mathcal{X}_{\mathbf{D}_n}$ and it is the degenerate subfamily in $\mathcal{X}_{\mathbf{D}_n}$ whose generic member is an affine log Calabi-Yau surface deformation equivalent to $U = Y \backslash D$ with du Val singularities. Moreover, the cluster complex in $\mathfrak{D}_{\mathbf{s}}$ break into two pieces inside $\mathfrak{D}_{\bar{\mathbf{s}}}$:

THEOREM 1.2.2. *(Theorem 3.4.6, Corollary 3.4.7) The scattering diagram $\mathfrak{D}_{\bar{\mathbf{s}}}$ has two disjoint subfans. Let $\tilde{\mathcal{A}}_{\mathrm{prin},\bar{\mathbf{s}}}$ be the corresponding log Calabi-Yau variety we build using all chambers in these two subfans. Then $\tilde{\mathcal{A}}_{\mathrm{prin},\bar{\mathbf{s}}}$ has two non-equivalent cluster structures.*

In the main example we investigate in Chapter 2, $\mathcal{X}_{\mathbb{S}_4^2}$ is isomorphic to $\mathcal{X}_{\mathbf{D}_4}$, the universal family of affine cubic surfaces and $\mathcal{X}_{\mathbb{T}_1^2}$ is isomorphic to the degenerate subfamily in $\mathcal{X}_{\mathbf{D}_4}$ whose generic member is a singular affine cubic surface with three ordinary double points. So Theorem 2.2 is a generalization of Theorem 1.2.

Moreover, by relating the Donaldson-Thomas transformations to reflections in the Weyl groups, we obtain the following theorem:

THEOREM 1.2.3. *(Theorem 3.4.4, subsection 3.4.3) The Donaldson-Thomas transformation of $\mathcal{X}_{D^\perp}$ for $D^\perp \simeq \mathbf{D}_n (n \geqslant 4)$ or $\mathbf{E}_n (n = 6, 7)$ is cluster, that is, it can be factored as compositions of cluster transformations.*

Thus, the action of Weyl group gives us a simple conceptual understanding of the DT-transformations. Due to the intricate nature of the Weyl group of $\mathbf{E}_8$, we leave it for future exploration.

CHAPTER 2

Cluster structures and subfans in the scattering diagrams

2.1. Introduction

2.1.1. Fock-Goncharov duality conjecture for $\mathcal{A}_{\mathbb{T}_1^2}, \mathcal{X}_{\mathbb{T}_1^2}$ and $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$.

In [**FG06**], Fock and Goncharov introduced cluster coordinates on moduli spaces of local systems on Riemann surfaces in Teichmüller theory. In this paper, the main examples we investigate are cluster varieties with seeds coming from ideal triangulations of once punctured torus $\mathbb{T}_1^2$. These varieties parametrize PGL_2-local systems with decorations or framing on $\mathbb{T}_1^2$, as constructed in [**FG06**]. Any ideal triangulation of $\mathbb{T}_1^2$ yields the same isomorphic class of seeds that can be encoded by Markov quiver:

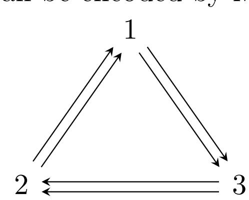

Figure 2.1.1. Markov quiver

Let $\mathcal{A}_{\mathbb{T}_1^2}$, $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ and $\mathcal{X}_{\mathbb{T}_1^2}$ be cluster varieties associated to $\mathbb{T}_1^2$ of type $\mathcal{A}$, $\mathcal{A}_{\mathrm{prin}}$ and $\mathcal{X}$ respectively. Using techniques of *scattering diagrams* (c.f. [**KS06**, **GS11**]) and *broken lines* (c.f. [**G09**, **CPS**]) coming from mirror symmetry, Gross, Hacking, Keel and Kontsevich in [**GHKK**] construct canonical bases for cluster varieties called *theta functions*, which are generalization of *cluster variables*. Specifically, given a cluster variety V, let $V^\vee$ be its *Fock-Goncharov dual* variety and $V^\vee(\mathbb{Z}^T)$ the set of integer tropical points of $V^\vee$. For each q in $V^\vee(\mathbb{Z}^T)$, there is a theta function ϑ_q corresponding to q. Theta functions can be formal power series. However, there is a canonically defined subset $\Theta(V)$ in $V^\vee(\mathbb{Z}^T)$ called *theta set* that index a vector subspace $\mathrm{mid}(V)$ of $\mathrm{can}(V)$ and $\mathrm{mid}(V)$ has a $\mathbb{C}$-algebra structure. Moreover, $\mathrm{mid}(V)$ can always be identified as a subring of $\Gamma(V, \mathcal{O}_V)$ if V is of type $\mathcal{X}$ or $\mathcal{A}_{\mathrm{prin}}$. In the case of cluster varieties associated to $\mathbb{T}_1^2$, it is known that scattering diagrams on $\mathcal{A}^\vee_{\mathbb{T}_1^2}(\mathbb{R}^T)$ and $\mathcal{A}^\vee_{\mathrm{prin},\mathbb{T}_1^2}(\mathbb{R}^T)$ have two disjoint subfan structures, one called *cluster complex* consisting of positive chambers $\{\mathcal{C}^+_{\overline{\mathbf{s}}_v}\}_{v\in\mathfrak{T}_{\overline{\mathbf{s}}}}$ and the other consisting of negative chambers $\{\mathcal{C}^-_{\overline{\mathbf{s}}_v}\}_{v\in\mathfrak{T}_{\overline{\mathbf{s}}}}$. Here $\mathfrak{T}_{\overline{\mathbf{s}}}$ is the infinite tree whose vertices parametrizing all the seeds mutationally equivalent to an initial seed $\overline{\mathbf{s}}$ coming from an ideal triangulation of $\mathbb{T}_1^2$. Because of the infinite wall-crossings from $\mathcal{C}^-_{\overline{\mathbf{s}}}$ to $\mathcal{C}^+_{\overline{\mathbf{s}}}$ in the scattering diagram, it is not clear whether theta functions parametrized by integer tropical points in $\mathcal{C}^-_{\overline{\mathbf{s}}}$ are formal sums or polynomials. Given a cluster variety V, we say that *Full Fock-Goncharov conjecture* holds for V if the following holds:

$$\Gamma(V, \mathcal{O}_V) = \mathrm{mid}(V) = \mathrm{can}(V) = \bigoplus_{q\in V^\vee(\mathbb{Z}^T)} \mathbb{C}\cdot\vartheta_q$$

In this paper, one of our main results is the following:

THEOREM 2.1.1. *(cf. Theorem 2.5.1, Corollary 2.5.2) The Full Fock-Goncharov conjecture holds for $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ and $\mathcal{X}_{\mathbb{T}_1^2}$.*

It is unnatural to build up $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ using only the atlas $\mathcal{T}^+$ of torus charts parametrized by positive chambers and exclude the atlas $\mathcal{T}^-$ of those parametrized by negative chambers. If we use torus charts in both $\mathcal{T}^+$ and $\mathcal{T}^-$, we will get a new variety $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ that contains the original cluster variety $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$. Since $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}\backslash\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ has codimension 2 in $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$, $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ and $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ have the same ring of regular functions. The atlas $\mathcal{T}^-$ also provides for $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ *an atlas of cluster torus charts* (c.f. Definition 2.5.3) that are not mutationally equivalent to those in $\mathcal{T}^+$. Therefore $\mathcal{T}^-$ equips $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ with a cluster structure non-equivalent the one provided by $\mathcal{T}^+$. See Definition 2.5.4 for what we precisely mean by two cluster structures are non-equivalent. Hence we have the following theorem:

THEOREM 2.1.2. *(cf. Theorem 2.5.7)The variety $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ is a log Calabi-Yau variety with 2 non-equivalent cluster structures, each of which corresponds to one of the two disjoint simplicial fans in the scattering diagram.*

The Full Fock-Goncharov conjecture says that $\mathcal{A}^\vee_{\mathbb{T}_1^2}(\mathbb{Z}^T)$ should parametrize a vector space basis for $\Gamma(\mathcal{A}_{\mathbb{T}_1^2}, \mathcal{O}_{\mathcal{A}_{\mathbb{T}_1^2}})$ and it turns out to be false.

THEOREM 2.1.3. *The Full Fock-Goncharov conjecture does not hold for $\mathcal{A}_{\mathbb{T}_1^2}$. We have*

$$\Theta(\mathcal{A}_{\mathbb{T}_1^2}) = \mathcal{A}^\vee_{\mathbb{T}_1^2}(\mathbb{Z}^T)$$

and therefore,

$$\mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2}) = \mathrm{mid}(\mathcal{A}_{\mathbb{T}_1^2})$$

But $\mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2})$ is strictly contained in $\Gamma(\mathcal{A}_{\mathbb{T}_1^2}, \mathcal{O}_{\mathbb{T}_1^2})$.

It is not surprising that $\mathcal{A}^\vee_{\mathbb{T}_1^2}(\mathbb{Z}^T)$ does not parametrize a vector space basis for $\Gamma(\mathcal{A}_{\mathbb{T}_1^2}, \mathcal{O}_{\mathcal{A}_{\mathbb{T}_1^2}})$, if we take into consideration that when we build $\mathcal{A}_{\mathbb{T}_1^2}$, we use only half of the scattering diagram on $\mathcal{A}^\vee_{\mathbb{T}_1^2}(\mathbb{Z}^T)$. From the mirror symmetry perspective, it will be more natural to expect that $\mathcal{A}^\vee_{\mathbb{T}_1^2}(\mathbb{Z}^T)$ should parametrize a basis for the ring of regular functions of the space that include torus charts of all chambers in the scattering diagram. We can run a construction similar to what we have done for $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ and get a variety $\tilde{\mathcal{A}}_{\mathbb{T}_1^2} \supset \mathcal{A}_{\mathbb{T}_1^2}$ by also including torus charts coming from negative chambers in the scattering diagram. The original cluster variety $\mathcal{A}_{\mathbb{T}_1^2}$ has codimension 1 inside $\tilde{\mathcal{A}}_{\mathbb{T}_1^2}$, so $\Gamma(\tilde{\mathcal{A}}_{\mathbb{T}_1^2}, \mathcal{O}_{\tilde{\mathcal{A}}_{\mathbb{T}_1^2}})$ is strictly contained in $\Gamma(\mathcal{A}_{\mathbb{T}_1^2}, \mathcal{O}_{\mathbb{T}_1^2})$. Moreover, we have the following theorem:

THEOREM 2.1.4. *We have*

$$\mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2}) = \Gamma(\tilde{\mathcal{A}}_{\mathbb{T}_1^2}, \mathcal{O}_{\tilde{\mathcal{A}}_{\mathbb{T}_1^2}})$$

So $\mathcal{A}^\vee_{\mathbb{T}_1^2}(\mathbb{Z}^T)$ *parametrize a vector space basis for the ring of regular functions of the enlarged variety* $\tilde{\mathcal{A}}_{\mathbb{T}_1^2}$ *.*

The above theorem justifies that $\widetilde{\mathcal{A}}_{\mathbb{T}_1^2}$ should be the correct mirror dual variety to $\mathcal{A}^\vee_{\mathbb{T}_1^2} = \mathcal{X}_{\mathbb{T}_1^2}$ since it is the space where the duality conjecture becomes true. At the end of the introduction, we will elaborate more on this point against the backdrop of the mirror symmetry program.

2.1.2. Technical framework: Quiver folding in the context of scattering diagrams. In section 2.5, instead of directly studying canonical bases for $\mathcal{A}_{\mathbb{T}_1^2}$, $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ or $\mathcal{X}_{\mathbb{T}_1^2}$ and their scattering diagrams, we relate them to those of cluster varieties associated to 4-punctured sphere $\mathbb{S}^2_4$. In order to do this, we introduce techniques of quiver-folding in cluster theory into scattering diagrams.

Given a seed $\mathbf{s}$, if a seed $\overline{\mathbf{s}}$ can obtained from $\mathbf{s}$ through a procedure called *folding*[1], then I prove the following relationship between the corresponding scattering diagrams:

THEOREM 2.1.5. *(cf. Construction 2.2.1, Theorem 2.2.19) The scattering diagram* $\mathfrak{D}_{\overline{\mathbf{s}}}$ *can be obtained from* $\mathfrak{D}_{\mathbf{s}}$ *via a quotient construction.*

The folding procedure can also be applied to theta functions. Given a cluster variety V, let $\Theta(V) \subset V^\vee(\mathbb{Z}^T)$ be the subset that parametrize theta functions that are contained in the ring of regular functions on V. Let $\mathcal{A}_{\mathrm{prin},\mathbf{s}}$ and $\mathcal{A}_{\mathrm{prin},\overline{\mathbf{s}}}$ be the cluster $\mathcal{A}_{\mathrm{prin}}$-variety with initial seed $\mathbf{s}$ and $\overline{\mathbf{s}}$ respectively. Then I prove the following theorem:

[1] In this paper, I only deal with a special class of folding, see Subsection 2.3.1

Theorem 2.1.6. *(cf. 2.3.6) If* $\Theta(\mathcal{A}_{\mathrm{prin},\mathbf{s}}) = \mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}}(\mathbb{Z}^T)$, *then* $\Theta(\mathcal{A}_{\mathrm{prin},\bar{\mathbf{s}}}) = \mathcal{A}^\vee_{\mathrm{prin},\bar{\mathbf{s}}}(\mathbb{Z}^T)$.

2.1.3. Building $\mathcal{A}_{\mathrm{prin}}$ using all chambers in the scattering diagram. Let $\mathbf{s}_0$ be an initial seed. Recall that the scattering diagram $\mathfrak{D}_{\mathbf{s}_0}$ always has two subfans, the cluster complex $\Delta^+_{\mathbf{s}_0}$ that consists of positive chambers and $\Delta^-_{\mathbf{s}_0}$ that consists of negative chambers. The two complexes are either disjoint or coincide. In [**GHKK**], a positive space $\mathcal{A}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ is built by attaching a copy of algebraic torus to each chamber in $\Delta^+_{\mathbf{s}_0}$ and gluing these tori together using the wall-crossings between these chambers. The positive space $\mathcal{A}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ is isomorphic to $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}$. In section 2.4, we build a positive space $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0} \supset \mathcal{A}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ using all chambers in $\Delta^+_{\mathbf{s}_0} \bigcup \Delta^-_{\mathbf{s}_0}$ if the path ordered product $\mathfrak{p}^{\mathbf{s}_0}_{+,-}$, or equivalently the Donaldson-Thomas transformation of $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}$ as defined in [**GS16**], is rational. Let $\mathcal{A}^{\mathrm{scat},-}_{\mathrm{prin},\mathbf{s}_0} \subset \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ be the subspace built by gluing all tori attached to negative chambers. We prove that $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ is isomorphic to a variety built by two copies of $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}$ together:

Theorem 2.1.7. *We have an isomorphism of positive spaces:*

$$\tilde{\mathcal{A}}_{\mathrm{prin},\mathbf{s}_0} = \mathcal{A}^+_{\mathrm{prin}\mathbf{s}_0} \bigcup \mathcal{A}^-_{\mathrm{prin},\mathbf{s}_0} \xrightarrow{\sim} \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$$

Here $\mathcal{A}^\pm_{\mathrm{prin},\mathbf{s}_0}$ *are two copies of* $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}$, *the image of* $\mathcal{A}^+_{\mathrm{prin},\mathbf{s}_0}$ *is* $\mathcal{A}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ *and the image of* $\mathcal{A}^-_{\mathrm{prin},\mathbf{s}_0}$ *is* $\mathcal{A}^{\mathrm{scat},-}_{\mathrm{prin},\mathbf{s}_0}$.

The construction of $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}^2_1}$ in the beginning is a specific instance of the above theorem. In general, $\mathfrak{D}_{\mathbf{s}_0}$ might have subfans other than $\Delta^+_{\mathbf{s}_0}$ and $\Delta^-_{\mathbf{s}_0}$. It will be interesting to produce such an example and see if these subfans can yield non equivalent cluster structures as in the case of $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}^2_1}$.

2.1.4. Background in mirror symmetry. In this subsection, we will recall some background in mirror symmetry and review our main results in the light of mirror symmetry. Given an affine log Calabi-Yau variety V over a ground field $\mathbb{K}$ of characteristic 0, inspired by the homological mirror symmetry conjecture, Gross, Hacking and Keel conjecture (Abouzaid, Kontsevich and Soibelman also suggest) that $\Gamma(V, \mathcal{O}_V)$, the ring of regular functions of V, has a canonical vector space basis called *theta functions* parametrized by the integer tropical points $V^\vee(\mathbb{Z}^T)$ on the mirror dual variety $V^\vee$. Moreover, the structure constants of $\Gamma(V, \mathcal{O}_V)$ as a $\mathbb{K}$-algebra can be computed by counting holomorphic discs $V^\vee$. For the development of mirror symmetry of log Calabi-Yau surfaces, see [**A07**, **GHK11**, **Yu16**, **Yu17**].

The subject of cluster algebra is invented to study Lusztig's *dual canonical basis* in the ring of regular functions on base affine space G/N where G is a reductive algebraic group and N an unipotent subgroup. As an attempt to find an algorithm to compute Lusztig's dual canonical basis, Fomin and Zelevinsky discovered that $\Gamma(G/N, \mathcal{O}_{G/N})$ has cluster structure and G/N is a cluster variety.

It turns out cluster varieties are particular examples of log Calabi-Yau varieties. Therefore mirror symmetry predicts a geometric way to produce a canonical basis for cluster varieties as mentioned in the beginning. The first attempt in this vein is [**GHKK**]. Let us briefly review the significance of [**GHKK**] towards cluster theory. First, though scattering diagrams and broken lines in [**GHKK**] are still combinatorial gadgets, mirror symmetry predicts that broken lines and theta functions should be intrinsic to the enumerative geometry on the mirror dual varieties and therefore independent of the auxiliary combinatorial data of cluster algebras. In fact, the technology has developed afterwards and in [**KY**], this is proved to be true for cluster varieties whose rings of regular functions are finitely generated and smooth.

It turns out that theta functions produced by combinatorial methods in [**FZ02**, **FZ07**], called *cluster variables*, are only parametrized by integer tropical points in the cluster complex in the scattering diagram. Cluster complex is a subfan structure in the scattering diagram. In some cases, we expect disjoint subfan structures in the scattering diagrams to yield non-equivalent cluster structures for cluster varieties. This paper produces a first example along this line, as mentioned in Theorem 2.1.2. The insight of [**GHKK**] also gives correction for the original Fock-Goncharov duality conjecture. The mirror dual variety to $\mathcal{X}$-variety should be the space built by the scattering diagram on $\mathcal{X}^\vee(\mathbb{Z}^T)$ which includes torus charts coming from all chambers in the scattering diagram. Since $\mathcal{A}$-variety includes only torus charts coming from chambers in the cluster complex, in general it will not be the correct mirror to $\mathcal{X}$-variety. Therefore the duality conjecture in general will not hold for $\mathcal{X}$- and $\mathcal{A}$-cluster varieties, as in the case of $\mathcal{X}_{\mathbb{T}_1^2}$ and $\mathcal{A}_{\mathbb{T}_1^2}$.

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2.2. A quotient construction of scattering diagrams

2.2.1. Scattering diagrams. In this subsection, we will briefly recall basics of scattering diagrams and prove lemmas we need for the rest of the paper. See [**GHKK**] chapter 1 for a more detailed exposition on scattering diagrams.

Let N be a lattice with dual lattice M and a skew-symmetric bilinear form

$$\{,\} : N \times N \to \mathbb{Q}$$

Let I be the index set $\{1, 2, \cdots, \mathrm{rank} N\}$. Fix a basis $(e_i)_{i \in I}$ for N. Define $N^+ \subset N$ as follows:

$$N^+ = \{\sum_{i \in I} a_i e_i \mid a_i \in \mathbb{N}, \sum_{i \in I} a_i > 0\}$$

Let $\mathfrak{g} = \bigoplus_{n \in N^+} \mathfrak{g}_n$ be a Lie algebra over a ground field $\mathbb{K}$ of characteristic 0. Assume that $\mathfrak{g}$ is graded by N^+, that is, $[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] \subset \mathfrak{g}_{n_1+n_2}$, and that $\mathfrak{g}$ is skew-symmetric for $\{,\}$, that is, $[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] = 0$ if $\{n_1, n_2\} = 0$. Define the following linear functional on N:

$$d : N \to \mathbb{Z}$$

$$\sum_{i \in I} a_i e_i \mapsto \sum_{i \in I} a_i$$

For each $k \in \mathbb{Z}_{\geqslant 0}$, there is an Lie subalgebra $\mathfrak{g}^{>k}$ of $\mathfrak{g}$:

$$\mathfrak{g}^{>k} = \bigoplus_{n \in N^+ | d(n) > k} \mathfrak{g}_n$$

And a N^+-graded nilpotent Lie algebra $\mathfrak{g}^{\leqslant k}$:

$$\mathfrak{g}^{\leqslant k} = \mathfrak{g}/\mathfrak{g}^{>k}$$

We denote the unipotent Lie group corresponding to $\mathfrak{g}^{\leqslant k}$ as $G^{\leqslant k}$. As a set, it is in bijection with $\mathfrak{g}^{\leqslant k}$, but the group multiplication is given by the Baker-Campbell-Hausdorff formula. Given $i < j$, we have canonical homomorphism:

$$\mathfrak{g}^{\leqslant j} \to \mathfrak{g}^{\leqslant i}, \quad G^{\leqslant j} \to G^{\leqslant i}$$

Define the pro-nilpotent Lie algebra and its corresponding pro-unipotent Lie group by taking the projective limits:

$$\hat{\mathfrak{g}} = \varprojlim \mathfrak{g}^{\leqslant k}, \quad G = \varprojlim G^{\leqslant k}$$

We have canonical bijections:

$$\exp : \mathfrak{g}^{\leqslant k} \to G^{\leqslant k}, \quad \exp : \hat{\mathfrak{g}} \to G$$

Moreover, given any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we can define a Lie subalgebra $\hat{\mathfrak{h}}$ of the completion $\hat{\mathfrak{g}}$ and its corresponding Lie subgroup $\exp(\hat{\mathfrak{h}}) \subset G$:

$$\hat{\mathfrak{h}} = \varprojlim \mathfrak{h}^{\leqslant k}, \quad \mathfrak{h}^{\leqslant k} = \mathfrak{h} / \left(\mathfrak{h} \cap \mathfrak{g}^{>k}\right)$$

Given an element n_0 in N^+, there is an Lie subalgebra $\mathfrak{g}_{n_0}^{||}$ of $\mathfrak{g}$:

$$\mathfrak{g}_{n_0}^{||} = \bigoplus_{k \in \mathbb{Z}_{>0}} \mathfrak{g}_{k \cdot n_0}$$

And its corresponding Lie subgroup of G:

$$G_{n_0}^{||} = \exp(\widehat{\mathfrak{g}_{n_0}^{||}})$$

By our assumption that $\mathfrak{g}$ is skew-symmetric for $\{,\}$, $\mathfrak{g}_{n_0}^{||}$ and therefore $G_{n_0}^{||}$ are abelian.

Definition 2.2.1. A *wall* in $M_{\mathbb{R}}$ for N^+ and $\mathfrak{g}$ is pair $(\mathfrak{d}, g_{\mathfrak{d}})$ such that

1) $g_{\mathfrak{d}}$ is contained in $G_{n_0}^{||}$ for some primitive element n_0 in N^+

2) $\mathfrak{d}$ is contained in $n_0^{\perp} \subset M_{\mathbb{R}}$ and is a convex rational polyhedral cone of dimension $(\mathrm{rank} N - 1)$.

For a wall $(\mathfrak{d}, g_{\mathfrak{d}})$, we say $\mathfrak{d}$ is its *support*.

Definition 2.2.2. A *scattering diagram* $\mathfrak{D}$ for N^+ and $\mathfrak{g}$ is a collection of walls such that for every order k in $\mathbb{Z}_{\geqslant 0}$, there are only finitely many walls

$(\mathfrak{d}, g_{\mathfrak{d}})$ in $\mathfrak{D}$ with non-trivial image of $g_{\mathfrak{d}}$ under the projection map $G \to G^{\leqslant k}$. Given a scattering diagram $\mathfrak{D}$ and an order $k \geqslant 0$, by passing the attached group elements $g_{\mathfrak{d}}$ through the projection $G \to G^{\leqslant k}$, we get a scattering diagram $\mathfrak{D}^{\leqslant k}$ for N^+ and $\mathfrak{g}^{\leqslant k}$.

REMARK 2.2.3. The finite condition in the definition of the scattering diagram enables us, up to order k, reduce to the case where the scattering diagram has only finitely many walls, a trick repeatedly used in this paper.

DEFINITION 2.2.4. We define the *essential support* of a scattering digram $\mathfrak{D}$ to be following set:

$$\mathrm{supp}_{\mathrm{ess}}(\mathfrak{D}) = \bigcup_{(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D} | g_{\mathfrak{d}} \neq 0} \mathfrak{d}$$

We define the *singular locus* of a scattering diagram $\mathfrak{D}$ to be the following set:

$$\mathrm{sing}(\mathfrak{D}) = \bigcup_{(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}} \partial \mathfrak{d} \cup \bigcup_{\substack{(\mathfrak{d}_1, g_{\mathfrak{d}_1}), (\mathfrak{d}_2, g_{\mathfrak{d}}) \in \mathfrak{D} \\ \dim(\mathfrak{d}_1 \cap \mathfrak{d}_2) = 2}} \mathfrak{d}_1 \cap \mathfrak{d}_2$$

DEFINITION 2.2.5. A point x in $M_{\mathbb{R}}$ is general if there exists at most one rational hyperplane $n_0^{\perp}$ such that x is contained in $n_0^{\perp}$. Given a general point x in $M_{\mathbb{R}}$, define

$$g_x(\mathfrak{D}) = \prod_{(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D} | \mathfrak{d} \ni x} g_{\mathfrak{d}}$$

DEFINITION 2.2.6. Two scattering diagrams $\mathfrak{D}, \mathfrak{D}'$ are *equivalent* if and only if $g_x(\mathfrak{D}) = g_x(\mathfrak{D}')$ for all general points x in $M_{\mathbb{R}}$.

Let $\gamma : [0, 1] \to \mathfrak{D}$ be a smooth immersion. We say that γ is generic for $\mathfrak{D}$ if it satisfies the following conditions:

i) The end points $\gamma(0)$ and $\gamma(1)$ does not lie in the essential support of $\mathfrak{D}$.

ii) γ does not pass through the singular locus of $\mathfrak{D}$

iii) Whenever γ crosses a wall, it crosses it transversally.

Then given a path γ generic for $\mathfrak{D}$, let $\mathfrak{p}_{\gamma,\mathfrak{D}}$ be the *path-ordered product* as defined in [**GHKK**].

LEMMA 2.2.7. *Two scattering diagram $\mathfrak{D}$ and $\mathfrak{D}'$ are equivalent if and only if $\mathfrak{p}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\gamma,\mathfrak{D}'}$ for any path γ that is generic for both $\mathfrak{D}$ and $\mathfrak{D}'$.*

DEFINITION 2.2.8. A scattering diagram $\mathfrak{D}$ is *consistent* if for any path γ that is generic for it, the path-ordered product $\mathfrak{p}_{\gamma,\mathfrak{D}}$ depends only on the end points of γ.

DEFINITION 2.2.9. Define the following chambers in $M_{\mathbb{R}}$:

$$\mathcal{C}^+ = \{m \in M_{\mathbb{R}} \mid m\,|_{N^+} \geqslant 0\}$$

$$\mathcal{C}^- = \{m \in M_{\mathbb{R}} \mid m\,|_{N^+} \leqslant 0\}$$

We call $\mathcal{C}^+$ and $\mathcal{C}^-$ the *positive chamber* and the *negative chamber* respectively.

THEOREM 2.2.10. *([**KS13**],[**GHKK**]) Given an element $\mathfrak{p}_{+,-}$ in G, there is an unique equivalence class of consistent scattering diagrams corresponding to $\mathfrak{p}_{+,-}$. Choose a representative $\mathfrak{D}_{\mathfrak{p}_{+,-}}$ in this equivalence class. The element $\mathfrak{p}_{+,-}$ is just the path-ordered product along any path with the initial point in the interior of $\mathcal{C}^+$ and the end point in the interior of $\mathcal{C}^-$.*

DEFINITION 2.2.11. Given a wall $(\mathfrak{d}, g_{\mathfrak{d}})$ such that $\mathfrak{d}$ is contained in $n_0^{\perp}$ for a primitive element $n_0 \in N^+$, we say that $(\mathfrak{d}, g_{\mathfrak{d}})$ is *incoming* if $\{n_0, \cdot\}$ is contained in $\mathfrak{d}$. Otherwise, we say that $(\mathfrak{d}, g_{\mathfrak{d}})$ is *outgoing*.

THEOREM 2.2.12. *([**GHKK**]) The equivalence class of a consistent scattering diagram is determined by its set of incoming walls.*

Given a primitive element n_0 in N^+, consider the following splitting of $\mathfrak{g}$ with respect to n_0:

$$\mathfrak{g} = \mathfrak{g}_+^{n_0} \bigoplus \left(\mathfrak{g}_{n_0}^{\|} \bigoplus \mathfrak{g}_{n_0}^{\perp}\right) \bigoplus \mathfrak{g}_-^{n_0} \tag{2.2.1}$$

$$\mathfrak{g}_+^{n_0} := \bigoplus_{\{n_0,n\}>0} \mathfrak{g}_n, \quad \mathfrak{g}_-^{n_0} := \bigoplus_{\{n_0,n\}<0} \mathfrak{g}_n, \quad \mathfrak{g}_{n_0}^{\|} := \bigoplus_{k\in\mathbb{N}^+} \mathfrak{g}_{kn_0}, \quad \mathfrak{g}_{n_0}^{\perp} := \bigoplus_{\substack{\{n_0,n\}=0\\ n\notin\mathbb{N}^+\cdot n_0}} \mathfrak{g}_n$$

The above splitting of $\mathfrak{g}$ induces unique factorization $g = g_+^{n_0} \circ \left(g_{n_0}^{\|} \circ g_{n_0}^{\perp}\right) \circ g_-^{n_0}$. Thus we can define the following map

$$\begin{aligned} \Psi : G &\to \prod_{n_0\in N^+\text{primitive}} G_{n_0}^{\|} \\ g &\mapsto \prod_{n_0\in N^+\text{primitive}} g_{n_0}^{\|} \end{aligned}$$

Proposition 2.2.13. *([GHKK]) The map Ψ is a bijection.*

2.2.2. A quotient construction of scattering diagrams. Let Π be a finite group that acts on I and satisfies the following condition: For any indices $i, j \in I$, for any π, π' in Π, we have

$$\{e_{\pi\cdot i}, e_{\pi'\cdot j}\} = \{e_i, e_j\}. \tag{2.2.2}$$

The group Π acts on the basis $(e_i)_{i\in I}$ via permutation of indices in I. Moreover, its action on $(e_i)_{i\in I}$ can be extended $\mathbb{Z}$-linearly to an action on the lattice N and $\mathbb{R}$-linearly to an action on $N_{\mathbb{R}}$. Given π in Π, let $\pi^* : M_{\mathbb{R}} \to M_{\mathbb{R}}$ be the automorphism dual to $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}$. Assume that Π also acts as Lie algebra automorphisms on $\mathfrak{g}$ compatibly with its action on N^+, that is, for any π in Π, for any n in N^+,

$$(2.2.3) \qquad \pi \cdot \mathfrak{g}_n = \mathfrak{g}_{\pi(n)}$$

Proposition 2.2.14. *Given π in Π, for any general point x in $M_{\mathbb{R}}$, we have*

$$\pi \cdot g_x(\mathfrak{D}_{\mathfrak{p}_{+,-}}) = g_{(\pi^{-1})^* \cdot x}(\mathfrak{D}_{\pi \cdot \mathfrak{p}_{+,-}})$$

Proof. It is clear that the action of Π on $M_{\mathbb{R}}$ maps general points to general points. Given a general point x in $M_{\mathbb{R}}$, we have the splitting of Lie algebra $\mathfrak{g}$ with respect to x: $\mathfrak{g} = \mathfrak{g}_+^x \oplus \mathfrak{g}_0^x \oplus \mathfrak{g}_-^x$. This splitting induces an unique factorization of $\mathfrak{p}_{+,-}$:

$$\mathfrak{p}_{+,-} = (\mathfrak{p}_{+,-})_+^x \circ (\mathfrak{p}_{+,-})_0^x \circ (\mathfrak{p}_{+,-})_-^x$$

where $(\mathfrak{p}_{+,-})_\pm^x$ is contained in $\exp(\hat{\mathfrak{g}}_\pm^x)$ and $(\mathfrak{p}_{+,-})_0^x$ is contained in $\exp(\hat{\mathfrak{g}}_0^x)$. Notice that $\pi \cdot (\mathfrak{g}_+^x) = \mathfrak{g}_+^{(\pi^{-1})^* \cdot x}$. Indeed, by condition 2.2.3,

$$\begin{aligned}
\mathfrak{g}_n \subseteq \mathfrak{g}_+^{(\pi^{-1})^* \cdot x} &\Longleftrightarrow \left\langle n, \left(\pi^{-1}\right)^* \cdot x \right\rangle > 0 \\
&\Longleftrightarrow \left\langle \pi^{-1} \cdot n, x \right\rangle > 0 \\
&\Longleftrightarrow \mathfrak{g}_{\pi^{-1} \cdot n} \subseteq \mathfrak{g}_+^x \\
&\Longleftrightarrow \pi^{-1} \cdot \mathfrak{g}_n \subseteq \mathfrak{g}_+^x \\
&\Longleftrightarrow \mathfrak{g}_n \subseteq \pi \cdot \left(\mathfrak{g}_+^x\right)
\end{aligned}$$

Similarly, we have $\pi \cdot (\mathfrak{g}_0^x) = \mathfrak{g}_0^{(\pi^{-1})^* \cdot x}$ and $\pi \cdot (\mathfrak{g}_-^x) = \mathfrak{g}_-^{(\pi^{-1})^* \cdot x}$. Hence

$$\pi \cdot \mathfrak{p}_{+,-} = \pi \cdot (\mathfrak{p}_{+,-})_+^x \circ \pi \cdot (\mathfrak{p}_{+,-})_0^x \circ \pi \cdot (\mathfrak{p}_{+,-})_-^x$$

is the unique factorization of $\pi \cdot \mathfrak{p}_{+,-}$ with respect to $(\pi^{-1})^* \cdot x$. Therefore,

$$(\pi \cdot \mathfrak{p}_{+,-})_0^{(\pi^{-1})^*(x)} = \pi \cdot (\mathfrak{p}_{+,-})_0^x$$

By Theorem 2.2.10,

$$g_x(\mathfrak{D}_{\mathfrak{p}_{+,-}}) = (\mathfrak{p}_{+,-})_0^x$$
$$g_{(\pi^{-1})^*(x)}(\mathfrak{D}_{\pi\cdot\mathfrak{p}_{+,-}}) = (\pi\cdot\mathfrak{p}_{+,-})_0^{(\pi^{-1})^*(x)}$$

Hence, we get the equality:

$$g_{(\pi^{-1})^*(x)}(\mathfrak{D}_{\pi\cdot\mathfrak{p}_{+,-}}) = \pi\cdot(\mathfrak{p}_{+,-})_0^x = \pi\cdot g_x(\mathfrak{D}_{\mathfrak{p}_{+,-}})$$

$\square$

COROLLARY 2.2.15. *Let G^Π be the subgroup of G invariant under the action of Π. Then given an element $\mathfrak{p}_{+,-}$ in G^Π, the set of walls in $\mathfrak{D}_{\mathfrak{p}_{+,-}}$ is invariant under the action of Π in the following sense:*

$$\pi\cdot g_x(\mathfrak{D}_{\mathfrak{p}_{+,-}}) = g_{(\pi^{-1})^*\cdot x}(\mathfrak{D}_{\mathfrak{p}_{+,-}})$$

Inspired by the above corollary, we can define an action of Π on the set of walls for N^+ and $\mathfrak{g}$ as follows:

$$\pi\cdot(\mathfrak{d}, g_{\mathfrak{d}}) := \big((\pi^{-1})^*(\mathfrak{d}), \pi\cdot g_{\mathfrak{d}}\big) \tag{2.2.4}$$

Observe that $(\pi^{-1})^*(n_0^\perp) = (\pi\cdot n_0)^\perp$ for any primitive element n_0 in N^+. Indeed,

$$\begin{aligned}\langle n_0, x\rangle = 0 &\Longleftrightarrow \langle n_0, \pi^*(\pi^{-1})^*x\rangle = 0\\ &\Longleftrightarrow \langle \pi\cdot n_0, (\pi^{-1})^*x\rangle = 0\end{aligned}$$

So if $\mathfrak{d}$ is contained in $n_0^\perp$ for a primitive element n_0 in N^+, $(\pi^{-1})^*(\mathfrak{d})$ will be contained in $(\pi\cdot n_0)^\perp$. Since $g_{\mathfrak{d}}$ is contained in $G_{n_0}^{\|}$ and $\pi\cdot G_{n_0}^{\|} = G_{\pi\cdot n_0}^{\|}$, we conclude that $\pi \cdot g_{\mathfrak{d}}$ is contained in $G^{\parallel}_{\pi \cdot n_0}$. So the action of Π on the set of walls is well-defined.

A natural question arises: How to characterize the elements in G^{Π}? The following lemma gives a characterization of elements in G^{Π} in terms of incoming walls of the corresponding scattering diagrams.

LEMMA 2.2.16. *Given $\mathfrak{p}_{+,-}$ in G, set*

$$\mathfrak{D}_{\text{in}} := \left\{ \left(n_0^{\perp}, \Psi(\mathfrak{p}_{+,-})_{n_0} \right) \mid n_0 \in N^+ \text{ primitive} \right\}$$

where

$$\Psi : G \to \prod_{n_0 \in N^+ \text{ primitive}} G^{\parallel}_{n_0}$$

is the canonical projection map in Proposition 2.2.13. Then $\mathfrak{p}_{+,-}$ is in G^{Π} if and only if $\mathfrak{D}_{\text{in}}$ is invariant subset under the action of Π on the set of walls (cf. 2.2.4).

PROOF. By Proposition 1.20 of [**GHKK**], Ψ is a bijection of sets. We claim that the projection map Ψ is compatible with the action of Π: For any π in Π, for any $\mathfrak{p}_{+,-}$ in G,

$$\Psi(\pi \cdot \mathfrak{p}_{+,-})_{\pi \cdot n_0} = \pi \cdot \Psi(\mathfrak{p}_{+,-})_{n_0} \tag{2.2.5}$$

Let

$$\mathfrak{p}_{+,-} = (\mathfrak{p}_{+,-})^{+}_{n_0} \circ (\mathfrak{p}_{+,-})^{\parallel}_{n_0} \circ (\mathfrak{p}_{+,-})^{\perp}_{n_0} \circ (\mathfrak{p}_{+,-})^{-}_{n_0}$$

be the unique factorization of $\mathfrak{p}_{+,-}$ induced by splitting of $\mathfrak{g}$ in 2.2.5. Observe that

$$\begin{aligned}\pi \cdot (\mathfrak{g}_{+}^{n_0}) &= \bigoplus_{\substack{n \in N^+ \\ \{n_0, \pi^{-1}(n)\} > 0}} \mathfrak{g}_n \\ &= \bigoplus_{\substack{n \in N^+ \\ \{\pi(n_0), \pi(\pi^{-1}(n))\} > 0}} \mathfrak{g}_n \\ &= \bigoplus_{\substack{n \in N^+ \\ \{\pi(n_0), n\} > 0}} \mathfrak{g}_n \\ &= \mathfrak{g}_{+}^{\pi(n_0)}\end{aligned}$$

Similarly, we have $\pi \cdot (\mathfrak{g}_{-}^{n_0}) = \mathfrak{g}_{-}^{\pi(n_0)}$, $\pi \cdot (\mathfrak{g}_{n_0}^{\parallel}) = \mathfrak{g}_{\pi(n_0)}^{\parallel}$ and $\pi \cdot (\mathfrak{g}_{n_0}^{\perp}) = \mathfrak{g}_{\pi(n_0)}^{\perp}$. Therefore,

$$\pi \cdot \mathfrak{p}_{+,-} = \pi \cdot (\mathfrak{p}_{+,-})_{+}^{n_0} \circ \pi \cdot (\mathfrak{p}_{+,-})_{n_0}^{\parallel} \circ \pi \cdot (\mathfrak{p}_{+,-})_{n_0}^{\perp} \circ \pi \cdot (\mathfrak{p}_{+,-})_{-}^{n_0}$$

is the unique factorization of $\pi \cdot \mathfrak{p}_{+,-}$ with respect to $\pi \cdot n_0$. By definition of Ψ, we have

$$\Psi(\pi \cdot \mathfrak{p}_{+,-})_{\pi \cdot n_0} = \pi \cdot (\mathfrak{p}_{+,-})_{n_0}^{\parallel} = \pi \cdot \Psi(\mathfrak{p}_{+,-})_{n_0}$$

Therefore, $\mathfrak{p}_{+,-}$ is contained in G^{Π} if and only if for all π in Π, for all primitive n_0 in N^+, the following is satisfied:

$$\Psi(\pi \cdot \mathfrak{p}_{+,-})_{\pi \cdot n_0} = \pi \cdot \Psi(\mathfrak{p}_{+,-})_{n_0} = \Psi(\mathfrak{p}_{+,-})_{\pi \cdot n_0} \tag{2.2.6}$$

Therefore

$$\begin{aligned}\pi \cdot \left(n_0^{\perp}, \Psi(\mathfrak{p}_{+,-})_{n_0}\right) &= \left((\pi^{-1})^*(n_0^{\perp}), \pi \cdot (\Psi(\mathfrak{p}_{+,-})_{n_0})\right) \\ &= \Big((\pi \cdot n_0)^{\perp}, \pi \cdot (\Psi(\mathfrak{p}_{+,-})_{n_0})\Big)\end{aligned}$$

Hence the equality 2.2.6 is equivalent to that for any π in Π and any primitive n_0 in N^+,

$$\pi \cdot \left(n_0^{\perp}, \Psi(\mathfrak{p}_{+,-})_{n_0}\right) = \left(\left(\pi \cdot n_0\right)^{\perp}, \Psi(\mathfrak{p}_{+,-})_{\pi\cdot n_0}\right)$$

which is equivalent to that $\left(\mathfrak{D}_{\mathfrak{p}_{+,-}}\right)_{\text{in}}$ is invariant under the action of Π. By Theorem 2.2.10, $\left(\mathfrak{D}_{\mathfrak{p}_{+,-}}\right)_{\text{in}}$ determines the equivalence class of the scattering diagrams corresponding to $\mathfrak{p}_{+,-}$. Hence we have the bijection of sets as stated in the lemma. $\square$

The action of Π on N enables us to construct a quotient lattice $\overline{N}$ with a basis $(e_{\Pi i})_{\Pi i \in \overline{I}}$ where $\overline{I}$ is the index set of orbits of I under the action of I. Let $q : N \twoheadrightarrow \overline{N}$ be the natural quotient homomorphism of lattices that sends e_i to $e_{\Pi i}$. We define the following skew-symmetric bilinear form $\{,\}$ on $\overline{N}$:

$$\{e_{\Pi i}, e_{\Pi j}\} = \{e_i, e_j\}$$

The condition 2.2.2 implies that the skew symmetric form on $\overline{N}$ is well-defined and the skew symmetric form on N is the pull-back of that on $\overline{N}$. The surjective homomorphism of lattices q induces an injective homomorphism between dual lattices $q^* : \overline{M} \hookrightarrow M$, which extends to an injective linear map $q^* : \overline{M}_{\mathbb{R}} \hookrightarrow M_{\mathbb{R}}$. Define

$$\overline{N}^+ = \{a_{\Pi i} e_{\Pi i} \mid a_{\Pi i} \in \mathbb{Z}_{\geqslant 0}, \sum_{\Pi i \in I'} a_{\Pi i} > 0\}$$

Notice that $q(N^+) = \overline{N}^+$. Define the following linear function:

$$\overline{d} : \overline{N} \to \mathbb{Z}$$

$$\sum_{\Pi i \in I'} a_{\Pi i} e_{\Pi i} \mapsto \sum_{\Pi i \in I'} a_{\Pi i}$$

Given an element n_0 in N^+, let $\overline{n_0}$ be its image in $\overline{N}^+$ under q. Let $\bar{\mathfrak{g}} = \bigoplus_{\overline{n}\in\overline{N}^+} \bar{\mathfrak{g}}_{\overline{n}}$ be a $\overline{N}^+$-graded Lie algebra skew-symmetric for $\{,\}$ on $\overline{N}$. Similar to $\mathfrak{g}$, for every k in $\mathbb{Z}_{>0}$, we define

$$\bar{\mathfrak{g}}^{>k} = \bigoplus_{\overline{n}\in\overline{N}^+|\bar{d}(\overline{n})>k} \bar{\mathfrak{g}}_{\overline{n}}$$

$$\bar{\mathfrak{g}}^{\leqslant k} = \bar{\mathfrak{g}}/\bar{\mathfrak{g}}^{>k}, \quad \overline{G}^{\leqslant k} = \exp(\bar{\mathfrak{g}}^{\leqslant k})$$
$$\widehat{\bar{\mathfrak{g}}} = \varprojlim \bar{\mathfrak{g}}^{\leqslant k}, \quad G = \exp(\widehat{\bar{\mathfrak{g}}}) = \varprojlim \overline{G}^{\leqslant k}$$

Suppose that we are given an order-preserving graded Lie algebra homomorphism $\tilde{q} : \mathfrak{g} \to \bar{\mathfrak{g}}$ that is compatible with $q : N \twoheadrightarrow \overline{N}$, that is, $\tilde{q}$ satisfies the following conditions:

$$\begin{aligned} &1)\ \tilde{q}^{-1}(\bar{\mathfrak{g}}^{>k}) = \mathfrak{g}^{>k} \\ (2.2.7) \quad &2)\ \tilde{q}(\mathfrak{g}^{\|}_{n_0}) \text{ is contained in } \bar{\mathfrak{g}}^{\|}_{\overline{n_0}} \text{ for any primitive element } n_0 \text{ in } N^+ \end{aligned}$$

Then for each order $k \in \mathbb{Z}_{>0}$, we have an induced Lie algebra homomorphism: $\tilde{q} : \mathfrak{g}^{\leqslant k} \to \bar{\mathfrak{g}}^{\leqslant k}$. Let $\tilde{q} : G^{\leqslant k} \to \overline{G}^{\leqslant k}$ be the corresponding homomorphism of Lie groups. Taking the limit of compositions of Lie group homomorphisms $G \to G^{\leqslant k} \to \overline{G}^{\leqslant k}$ for all $k \in \mathbb{Z}_{>0}$, we get a Lie group homomorphism $\tilde{q} : G \to \overline{G}$.

Let $\bar{n}_0$ be a primitive element in $\overline{N}^+$ and x a general element in $\bar{n}_0^{\perp}$. Notice that $q^*(x)$ would no longer necessarily be general in $M_{\mathbb{R}}$. However, we have the following lemma.

Lemma 2.2.17. *The direct sum*

$$\mathfrak{g}_0^{q^*(x)} = \bigoplus_{\substack{n\in N^+ \\ <n,q^*(x)>=0}} \mathfrak{g}_n$$

is an abelian Lie subalgebra.

Proof. Since $x \in \bar{n}_0^{\perp}$ is general, we have

$$\mathfrak{g}_0^{q^*(x)} = \bigoplus_{\substack{q(n)=k\overline{n}_0 \\ k\in\mathbb{N} \\ n\in N^+ \text{ is primitive}}} \mathfrak{g}_n^{\parallel}$$

For any primitive elements n_1, n_2 in N^+ such that $q(n_i) = k_i\overline{n}_0$, $k_i \in \mathbb{N}$,

$$\begin{aligned} \{n_1, n_2\} &= \{q(n_1), q(n_2)\} \\ &= \{k_1\overline{n}_0, k_2\overline{n}_0\} \\ &= 0 \end{aligned}$$

Since $\mathfrak{g}$ is skew symmetric for $\{,\}$ on N, $\{n_1, n_2\} = 0$ implies that $[\mathfrak{g}_{n_1}, \mathfrak{g}_{n_2}] = 0$. Therefore $\mathfrak{g}_0^{q^*(x)}$ is an abelian Lie subalgebra. $\square$

Lemma 2.2.18. *There exists a scattering diagram $\mathfrak{D}'$ in the equivalence class of $\mathfrak{D}_{\mathfrak{p}_{+,-}}$ such that for each wall $(\mathfrak{d}, g_{\mathfrak{d}})$ in $\mathfrak{D}'$ whose support contains $q^*(x)$ for a generic point x in $\overline{M}_{\mathbb{R}}$, $q^*(x)$ is contained in the interior of $\mathfrak{d}$.*

Proof. By Kontsevich-Soibelman's inductive algorithm to construct consistent scattering diagrams, $\mathfrak{D}_{\mathfrak{p}_{+,-}}$ is equivalent to a scattering diagram $\mathfrak{D}'$ whose walls all have boundaries contained in joints. Moreover, no boundary of any wall in $\mathfrak{D}'$ is contained in an *abelian joint*, that is, it is the intersection of two walls whose attached Lie group elements commute (cf. Definition-Lemma C.2., Lemma C.6 and Lemma C.7 of [**GHKK**]). Therefore, given a generic point x in $\overline{M}_{\mathbb{R}}$, assume that $q^*(x)$ is contained in the boundary of a wall in $\mathfrak{D}'$, $q^*(x)$ must be contained in a joint in $\mathfrak{D}'$ that is not abelian, which is in contradiction with Lemma 2.2.17. Hence the assumption is false, $q^*(x)$ is contained in the interior of any wall whose support contains it. $\square$

Construction 2.2.1. By Lemma 2.2.17, given any generic point x in $\overline{M}_{\mathbb{R}}$, if $q^*(x)$ is contained in the support of a wall $(\mathfrak{d}, g_{\mathfrak{d}})$ in $\mathfrak{D}_{\mathfrak{p}_{+,-}}$, we could assume that $q^*(x)$ is contained in the interior of $\mathfrak{d}$. Under this assumption, we build a scattering diagram $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$ in $\overline{M}_{\mathbb{R}}$. To do so, it suffices to build $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$ up to each order k. Modulo $G^{>k}$, there are only finitely many nontrivial walls in $\mathfrak{D}_{\mathfrak{p}_{+,-}}$. For each such nontrivial wall $(\mathfrak{d}, g_{\mathfrak{d}})$, if $\mathfrak{d} \cap q^*(\overline{M}_{\mathbb{R}})$ has codimension 1 in $q^*(\overline{M}_{\mathbb{R}})$, define $\overline{\mathfrak{d}} = q^{*-1}\left(\mathfrak{d} \cap q^*(\overline{M}_{\mathbb{R}})\right)$. Then $(\overline{\mathfrak{d}}, \overline{g}_{\mathfrak{d}})$ will be a wall for $\overline{N}^+$ and $\overline{\mathfrak{g}}$. Indeed, if $\mathfrak{d}$ is contained in $n^{\perp}$ for a primitive element n in N^+, then $\overline{n}$ is an integer multiple of $\overline{n}_0$ for a primitive element $\overline{n}_0$ in $\overline{N}^+$. Then $\overline{\mathfrak{d}}$ is contained in $\overline{n}_0^{\perp}$ and $\overline{g}_{\mathfrak{d}}$ is contained in $\overline{G}_{\overline{n}_0}$ by 2.2.7.

Let $\overline{\mathfrak{p}_{+,-}}$ be the image of $\mathfrak{p}_{+,-}$ under $\tilde{q} : G \to \overline{G}$. Let $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}}}$ be a consistent scattering diagram for $\overline{N}^+$ and $\overline{\mathfrak{g}}$ corresponding to $\overline{\mathfrak{p}_{+,-}}$. In the next proposition, we show that the quotient construction is compatible with the Lie group homomorphism $\tilde{q} : G \to \overline{G}$, that is, $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$ is equivalent to $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}}}$:

Theorem 2.2.19. *The scattering diagram $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$ is consistent and it belongs to the equivalence class of consistent scattering diagrams uniquely determined by $\overline{\mathfrak{p}_{+,-}}$.*

Proof. It suffices to show that for each order k, $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$ is consistent and is equivalent to $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}}}$ modulo $\overline{G}^{>k}$. To prove the consistency of $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$, given a generic path $\overline{\gamma}$ in $\overline{M}_{\mathbb{R}}$, we want to show that the path-ordered product $\mathfrak{p}_{\overline{\gamma}, \overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}}$ depends only on end points of $\overline{\gamma}$. We show that we could always perturb $q^*(\overline{\gamma})$ to a generic path γ in $M_{\mathbb{R}}$ with the same end points such that

$$\mathfrak{p}_{\overline{\gamma}, \overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}} = \overline{\mathfrak{p}_{\gamma, \mathfrak{D}_{\mathfrak{p}_{+,-}}}}$$

Let $0 < t_1 < \cdots < t_l < 1$ be time numbers when $\overline{\gamma}$ crosses a wall in $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$. For any $t \notin \{t_1, \cdots, t_l\}$, we claim that $q^*\left(\overline{\gamma}(t)\right)$ is not contained in any nontrivial wall in $\mathfrak{D}_{\mathfrak{p}_{+,-}}$. Indeed, the intersection of the support of a wall in $\mathfrak{D}_{\mathfrak{p}_{+,-}}$ with $q^*(\overline{M}_{\mathbb{R}})$ is either empty or has codimension 1 in $q^*(\overline{M}_{\mathbb{R}})$. Thus if $q^*\left(\overline{\gamma}(t)\right)$ is contained in a non-trivial wall in $\mathfrak{D}_{\mathfrak{p}_{+,-}}$, $\overline{\gamma}(t)$ will cross a wall in $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$, which is in contradiction with that $t \notin \{t_1, \cdots, t_l\}$. Given $i \in \{1, \cdots l\}$, without loss of generality, we could assume that $< \overline{\gamma}^{'}(t_i), \overline{n_0} >< 0$ where n_0 is a primitive element in N^+ such that $\overline{n_0}^{\perp}$ is the hyperplane in $\overline{M}_{\mathbb{R}}$ that $\overline{\gamma}$ crosses at time t_i. Let $D = \{(\mathfrak{d}_1, g_{\mathfrak{d}_1}), \cdots, (\mathfrak{d}_j, g_{\mathfrak{d}_j})\}$ be the collection of all nontrivial walls in $\mathfrak{D}_{\mathfrak{p}_{+,-}}$ whose support contains $q^*(\overline{\gamma}(t))$ and whose intersection with $q^*(\overline{M}_{\mathbb{R}})$ has codimension 1 in $q^*(\overline{M}_{\mathbb{R}})$. For each $h \in \{1, \cdots, j\}$, let n_h be the primitive element in N^+ such that $\mathfrak{d}_h$ is contained in $n_h^{\perp}$. Let $\delta = \min\{t_i - t_{i-1}, t_{i+1} - t_i\}$. There exists $\epsilon \in (0, \delta)$ such that we can find a semi-circle C_i that has end points $q^*(\overline{\gamma}(t_i - \epsilon))$ and $q^*(\overline{\gamma}(t_i + \epsilon))$, crosses walls in D transversally and does not cross any non-trivial wall in $\mathfrak{D}_{\mathfrak{p}_{+,-}} \backslash D$. It is possible to find such a semi-circle since $q^*(\gamma(t_i))$ is contained in the interior of each wall in D and $\overline{\gamma}$ crosses no wall on $[t_i - \epsilon, t_i) \cup (t_i, t_i + \epsilon]$. Replace $q^*(\overline{\gamma})\mid_{[t_i-\epsilon, t_i+\epsilon]}$ with C. Then

$$\overline{g}_{\gamma(t_i)}(\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}) = \overline{\mathfrak{p}_{C_i, \mathfrak{D}_{\mathfrak{p}_{+,-}}}}$$

Repeat the perturbation for each time t_i , we get a generic path γ with the same end points as $\overline{\gamma}$ for $\mathfrak{D}_{\mathfrak{p}_{+,-}}$ such that $\mathfrak{p}_{\overline{\gamma}, \overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}} = \overline{\mathfrak{p}_{\gamma, \mathfrak{D}_{\mathfrak{p}_{+,-}}}}$. The consistency of $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$ thus follows from that of $\mathfrak{D}_{\mathfrak{p}_{+,-}}$. If we let $\overline{\gamma}$ be the path from $\overline{\mathcal{C}}^+$ to $\overline{\mathcal{C}}^-$ where $\overline{\mathcal{C}}^{\pm}$ are the positive chamber and the negative chamber in $\overline{M}_{\mathbb{R}}$, we see that $\overline{\mathfrak{D}_{\mathfrak{p}_{+,-}}}$ is equivalent to $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}}}$. □

Corollary 2.2.20. *Let x be a general point in $\overline{M}_{\mathbb{R}}$, then*

$$\overline{g}_x(\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}}}) = \prod_{\substack{(\mathfrak{d}, g_{\mathfrak{d}}) \in \mathfrak{D}_{\mathfrak{p}_{+,-}} \\ \mathfrak{d} \ni q^*(x)}} \overline{g_{\mathfrak{d}}}$$

where $\overline{g_{\mathfrak{d}}}$ is the image of $g_{\mathfrak{d}}$ under the composition $q : G \to \overline{G}$.

2.3. Apply the quotient construction to cluster scattering diagrams

In this section, we will apply techniques developed in the previous section to scattering diagrams arising from cluster theory. Given a skew-symmetric seed $\mathbf{s}$, via exploiting the symmetry of $\mathbf{s}$, we will construct a new seed $\bar{\mathbf{s}}$. Readers familiar with folding in cluster theory will recognize our construction as a special class of the folding procedure. The new contribution of this paper to folding techniques is the application to scattering diagrams. For previous work on folding, see [**Du08**, **FWZ**].

2.3.1. Application of folding to scattering diagrams. Fix a lattice N of rank r with a skew-symmetric $\mathbb{Z}$-bilinear form $\{,\} : N \times N \to \mathbb{Z}$ and an index set $I = \{1, 2, \cdots, r\}$. A *seed* for $(N, \{\cdot,\cdot\})$ is a labelled collection of elements of N:

$$\mathbf{s} := (e_i \mid i \in I)$$

Given a seed $\mathbf{s}$, define

$$N^+ = \{\sum_{i \in I} a_i e_i \mid a_i \in \mathbb{N}, \sum_{i \in I} a_i > 0\}$$

Remark 2.3.1. For simplicity, we assume that we do not have frozen variables for the entire paper and that the original seed $\mathbf{s}$ is simply laced, i.e., all $d_i = 1$. Later in this section, given a group Π acting on I, we will construct a new seed $\bar{\mathbf{s}}$ which will still have no frozen variables but might no longer be simply laced.

From now on, we always assume that we are working over the ground field $\mathbb{C}$. Let us recall how cluster theory fits into the framework in the previous section. Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice of N. Define the following algebraic torus over $\mathbb{C}$:

$$T_M = M \otimes_{\mathbb{Z}} \mathbb{G}_m = \mathrm{Hom}(N, \mathbb{G}_m) = \mathrm{Spec}(\mathbb{C}[N])$$

The skew-symmetric form $\{,\}$ on N gives the algebraic torus T_M a Poisson structure:

$$\{z^n, z^{n'}\} = \{n, n'\} z^{n+n'}$$

Define $\mathfrak{g} = \bigoplus_{n \in N^+} \mathbb{C} \cdot z^n$. The Poisson bracket $\{,\}$ on $\mathbb{C}[N]$ endows $\mathfrak{g}$ with a Lie bracket $[,]$: $[f,g] = -\{f,g\}$. Given n, n' in N^+ , if $\{n, n'\} = 0$, then $\{z^n, z^{n'}\} = 0$. Thus $\mathfrak{g}$ is a N^+-graded Lie algebra skew-symmetric for $\{,\}$ on N. Let $\hat{\mathfrak{g}}$ be the pro-nilpotent Lie algebra after completing $\mathfrak{g}$. Let $G = \exp(\hat{\mathfrak{g}})$ be the pro-unipotent Lie group corresponding to $\hat{\mathfrak{g}}$. The Lie group G acts on $\mathbb{C}[[z^{e_1}, \cdots, z^{e_r}]]$ via the Hamiltonian flow. Explicitly, an element f in $\mathfrak{g}$ acts on $\mathbb{C}[[z^{e_1}, \cdots, z^{e_r}]]$ via the vector field $\{, f\}$.

The seed $\mathbf{s}$ gives rises to a scattering diagram $\mathfrak{D}_{\mathbf{s}}$ whose set of initial walls are defined as follows:

$$\mathfrak{D}_{\mathbf{s},\mathrm{in}} = \left\{ (e_i^{\perp}, \exp\left(-\mathrm{Li}_2(-z^{e_i})\right)) \mid i \in I \right\}$$

where Li_2 is the dilogarithm function $\mathrm{Li}_2(x) = \sum_{k \geqslant 1} \frac{x^k}{k^2}$. Explicitly, $\exp\left(-\mathrm{Li}_2(-z^{e_i})\right)$ acts as the following automorphism:

$$z^n \mapsto z^n (1 + z^{e_i})^{\{n, e_i\}}$$

This is the coordinate free expression for the inverse of $\mathcal{X}$-cluster mutation.

Let Π be a group action on I such that for any indices $i, j \in I$, for any π_1, π_2 in Π,

$$\{e_i, e_j\} = \{e_{\pi_1 \cdot i}, e_{\pi_2 \cdot j}\} \tag{2.3.1}$$

The group Π acts naturally on N as follows:

$$\pi \cdot \sum_{i \in I} a_i e_i = \sum_{i \in I} a_i e_{\pi \cdot i}$$

The condition 2.3.1is to guarantee that the skew-symmetric pairing is independent of choice of representative in the orbit of Π acting on N. Let $\bar{I}$ be the set of orbits Πi of I under the action of Π. Consider the lattice $\overline{N}$ with a basis $\{e_{\Pi i}\}_{\Pi i \in \overline{I}}$ indexed by $\bar{I}$. There is a natural quotient homomorphism of lattices $q : N \to \overline{N}$ that sends e_i to $e_{\Pi i}$.

Define a $\mathbb{Z}$-valued skew-symmetric bilinear form $\{,\}$ on $\overline{N}$ as follows:$\{e_{\Pi i}, e_{\Pi j}\} = \{e_i, e_j\}$. By condition 2.3.1, the skew-symmetric form is well-defined. Now we have the following fixed data:

- The lattice $\overline{N}$ with the $\mathbb{Z}$-valued skew-symmetric bilinear form $\{,\}$.
- The index set $\overline{I}$ parametrizing the set of orbits of I under the action of Π.
- For each $\Pi i \in \overline{I}$, a positive integer $d_{\Pi i} = |q^{-1}\{e_{\Pi i}\}|$.

Define the following seed $\bar{\mathbf{s}} = (e_{\Pi i} \mid \Pi i \in \bar{I})$. Unlike [**GHKK**], we do not impose the condition that the greatest common divisor of $\{d_{\Pi i}\}_{\Pi i \in \overline{I}}$ need to be 1. Notice that the new seed $\bar{\mathbf{s}}$ may no longer be simply laced since $d_{\Pi i}$ may not be 1.

Take

$$\overline{N}^+ = \{\sum_{\Pi i \in \bar{I}} a_{\Pi i} e_{\Pi i} \mid a_{\Pi i} \geqslant 0, \sum a_{\Pi i} > 0\}$$

Let $\bar{\mathfrak{g}} \subset \mathbb{K}[\overline{N}]$ be the $\mathbb{C}$-vector space with basis $z^{\bar{n}}, \bar{n} \in \overline{N}^+$. Define a Lie bracket on $\bar{\mathfrak{g}}$ as follows:

$$[z^{e_{\Pi i}}, z^{e_{\Pi j}}] = -\{e_{\Pi i}, e_{\Pi j}\} z^{e_{\Pi i} + e_{\Pi j}}$$

Define $\tilde{q} : \mathfrak{g} \to \overline{\mathfrak{g}}$ by $z^n \mapsto z^{\overline{n}}$ where $\bar{n}$ is the image of n under the quotient homomorphism $q : N \to \overline{N}$. It is clear that $\tilde{q}$ is surjective. We check that $\tilde{q}$

is a Lie-algebra homomorphism. Indeed,

$$[\tilde{q}\,(z^n)\,, \tilde{q}(z^{n'})] = -\{\overline{n}, \overline{n'}\} z^{\overline{n+n'}} = -\{n, n'\} z^{\overline{n+n'}} = \tilde{q}\left([z^n, z^{n'}]\right)$$

For each order k, by our definition of $d : N^+ \to \mathbb{Z}$ and $\overline{d} : \overline{N}^+ \to \mathbb{Z}$, $\tilde{q}$ restricts to a Lie algebra homomorphism

$$\mathfrak{g}^{\leqslant k} = \mathfrak{g}/\mathfrak{g}^{>k} \to \overline{\mathfrak{g}}/\overline{\mathfrak{g}}^{>k} = \overline{\mathfrak{g}}^{\leqslant k}$$

which induces a surjective Lie group homomorphism $G^{\leqslant k} \to \overline{G}^{\leqslant k}$. Let $\tilde{q}^k : G \to \overline{G}^{\leqslant k}$ be the composition of projection map $G \to G^{\leqslant k}$ with the above homomorphism. Then we obtain the following Lie group homomorphism $\tilde{q} := \varprojlim \tilde{q}^k : G \to \overline{G}$. It is easy to check that both $\tilde{q}$ and the action of Π are compatible with the quotient homomorphism $N \to \overline{N}$ (cf. 2.2.3,2.2.7).

The seed $\overline{\mathbf{s}}$ gives rises to a scattering diagram $\mathfrak{D}_{\overline{\mathbf{s}}}$ whose set of incoming walls are defined as follows:

$$\mathfrak{D}_{\overline{\mathbf{s}},\mathrm{in}} = \left\{ (e_{\Pi i}^{\perp}, \exp\left(-d_{\Pi i} \mathrm{Li}_2(-z^{e_{\Pi i}})\right) \mid \Pi i \in I \right\}$$

Let $\mathfrak{p}_{+,-}^{\mathbf{s}}$ be the unique group element corresponding to $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{p}_{+,-}^{\overline{\mathbf{s}}}$ that of $\mathfrak{D}_{\overline{\mathbf{s}}}$. We show that image of $\mathfrak{p}_{+,-}^{\mathbf{s}}$ under the Lie group homomorphism $\tilde{q} : G \to \overline{G}$ is $\mathfrak{p}_{+,-}^{\overline{\mathbf{s}}}$ so that results proved in Section 2.2 can be applied to $\mathfrak{D}_{\mathbf{s}}$ and $\mathfrak{D}_{\overline{\mathbf{s}}}$.

THEOREM 2.3.2. *The scattering diagram $\mathfrak{D}_{\overline{\mathbf{s}}}$ is equivalent to $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}^{\mathbf{s}}}}$.*

PROOF. Let $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}^{\mathbf{s}}},\mathrm{in}}$ be the set of incoming walls of $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}^{\mathbf{s}}}}$. Given $(\overline{\mathfrak{d}}, g_{\overline{\mathfrak{d}}})$ in $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}^{\mathbf{s}}}}$ with $\overline{\mathfrak{d}}$ contained in $\overline{n_0}^{\perp}$ for a primitive element in N^+, $(\overline{\mathfrak{d}}, g_{\overline{\mathfrak{d}}})$ is contained in $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}^{\mathbf{s}}},\mathrm{in}}$ if and only if $\{\overline{n_0}, \cdot\}$ is contained in $\overline{\mathfrak{d}}$. Let $(\mathfrak{d}, g_{\mathfrak{d}})$ be the wall in $\mathfrak{D}_{\mathbf{s}}$ such that $q^*(\overline{\mathfrak{d}}) = \mathfrak{d} \bigcap \overline{M}_{\mathbb{R}}$ and $\overline{g_{\mathfrak{d}}} = g_{\overline{\mathfrak{d}}}$. Then $\{\overline{n_0}, \cdot\}$ is contained in $\overline{\mathfrak{d}}$ if and only if $q^*(\{\overline{n_0}, \cdot\})$ is contained in $q^*(\overline{\mathfrak{d}})$, which is equivalent to that $\{n_0, \cdot\}$ is contained in $\mathfrak{d} \bigcap \overline{M}_{\mathbb{R}}$. Hence $(\mathfrak{d}, g_{\mathfrak{d}})$ is contained in $\mathfrak{D}_{\mathbf{s},\mathrm{in}}$. Therefore each wall in $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}},\mathrm{in}}$ comes from a wall in $\mathfrak{D}_{\mathbf{s},\mathrm{in}}$. Given $(e_i^{\perp}, \exp\left(-\mathrm{Li}_2(-z^{e_i})\right))$ in $\mathfrak{D}_{\mathbf{s},\mathrm{in}}$,

$$q^{*-1}\left[e_i^{\perp}\bigcap q^*\left(\overline{M}_{\mathbb{R}}\right)\right] = e_{\Pi i}^{\perp}$$

So

$$\begin{aligned}
\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}},\mathrm{in}} &= \left\{\left(e_{\Pi i}^{\perp}, \overline{\exp\left(-\mathrm{Li}_2(-z^{e_i})\right)}\right) \mid i \in I\right\} \\
&= \left\{\left(e_{\Pi i}^{\perp}, \exp\left(-\mathrm{Li}_2(-z^{e_{\Pi i}})\right)\right) \mid i \in I\right\}
\end{aligned}$$

Therefore up to splitting of incoming walls in $\mathfrak{D}_{\overline{\mathbf{s}}}$, $\mathfrak{D}_{\overline{\mathfrak{p}_{+,-}},\mathrm{in}} = \mathfrak{D}_{\overline{\mathbf{s}},\mathrm{in}}$. By Theorem 2.2.12, $\mathfrak{D}_{\overline{\mathbf{s}}}$ and $\mathfrak{D}_{\overline{\mathfrak{p}^{\mathbf{s}}_{+,-}}}$ are equivalent. □

2.3.2. An illustrative example: A_3 folded to B_2. Let $I = \{1, 2, 3\}$. Let N be a lattice of rank 3 with a basis e_1, e_2, e_3 and a $\mathbb{Z}$-bilinear skew-symmetric form $\{,\}$ such that

$$\{e_1, e_2\} = \{e_3, e_2\} = -1, \quad \{e_1, e_3\} = 0$$

The labelled basis $(e_i)_{i\in I}$ defines a seed $\mathbf{s}$ of type A_3. It has exchange matrix

$$\epsilon = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Let $\Pi = \mathbb{Z}_2$ act on I by interchanging the index 1 and 3. Clearly the action of Π satisfies the conditions 2.3.1, so we can run our construction. The quotient lattice $\overline{N}$ is of rank 2 with a basis $e_{\Pi 1}, e_{\Pi 2}$. The $\mathbb{Z}$-linear skew symmetric form on $\overline{N}$ is defined as follows:

$$\{e_{\Pi 1}, e_{\Pi 2}\} = -1$$

The sublattice $\overline{N}^{\circ} \subset \overline{N}$ has a basis $d_{\Pi 1}e_{\Pi 1}, e_{\Pi 2}$ where $d_{\Pi 1} = 2$. The labelled basis $(e_{\Pi i})_{\Pi i \in \overline{I}}$ defines a seed $\overline{\mathbf{s}}$ of type B_2 with exchange matrix

$$\overline{\epsilon} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

The group Π acts on $\mathfrak{D}_{\mathbf{s}}$ in such a way that on the level of the support of $\mathfrak{D}_{\mathbf{s}}$, its action is reflection with respect to the plane $\{x - z = 0\} \subseteq M_{\mathbb{R}}$. The quotient map $q : N_{\mathbb{R}} \twoheadrightarrow \overline{N}_{\mathbb{R}}$ induces an embedding on the dual spaces $q^* : \overline{M}_{\mathbb{R}} \hookrightarrow M_{\mathbb{R}}$ that embed $\mathfrak{D}_{\overline{\mathbf{s}}}$ into the plane $\{x = z\}$. Through this embedding, we could recover $\mathfrak{D}_{\overline{\mathbf{s}}}$ via walls in $\mathfrak{D}_{\mathbf{s}}$ whose intersection with the plane $\{x - z = 0\}$ is dimension 1. We can see this immediately on the level of initial walls, observing that

$$q^*(e_{\Pi 1}^{\perp}) = e_1^{\perp} \bigcap \{x = z\} = e_3^{\perp} \bigcap \{x = z\} = e_1^{\perp} \cap e_3^{\perp}$$

$$\exp\left(-d_{\Pi 1}\mathrm{Li}_2\left(-z^{e_{\Pi 1}}\right)\right) = \overline{\exp\left(-\mathrm{Li}_2\left(-z^{e_1}\right)\right)\exp\left(-\mathrm{Li}_2\left(-z^{e_3}\right)\right)}$$

and

$$q^*(e_{\Pi 2}^{\perp}) = e_2^{\perp} \bigcap \{x = z\}$$

$$\exp\left(-\mathrm{Li}_2\left(-z^{e_{\Pi 2}}\right)\right) = \overline{\exp\left(-\mathrm{Li}_2\left(-z^{e_2}\right)\right)}$$

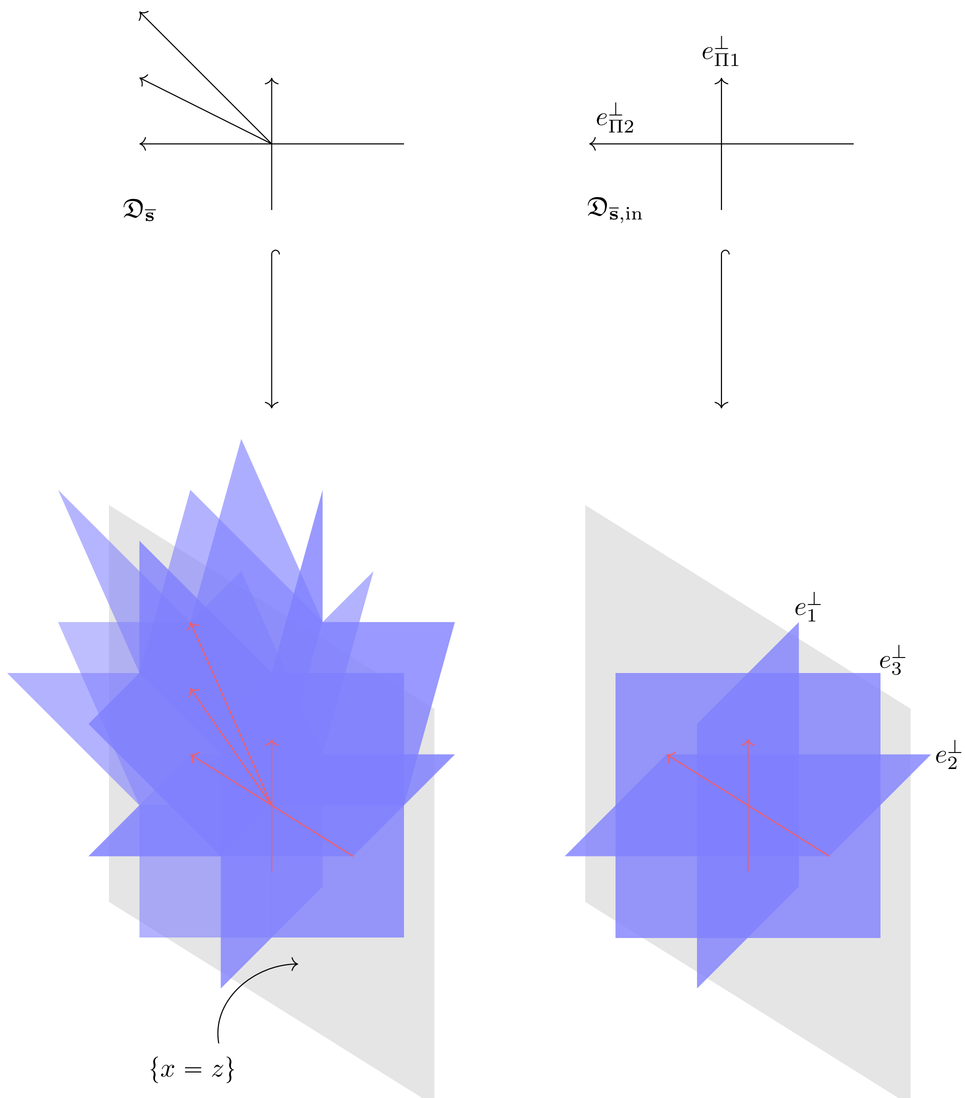


Figure 2.3.1. Embedding of scattering diagram $\mathfrak{D}_{\overline{\mathbf{s}}}$ of type B_2 into $\mathfrak{D}_{\mathbf{s}}$ of type A_3

2.3.3. Application of folding to theta functions. In this subsection, we want to apply folding techniques to canonical bases for $\mathcal{A}_{\mathrm{prin}}$-cluster varieties. Let $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{\mathrm{prin}}}$ be the scattering diagram for the $\mathcal{A}_{\mathrm{prin}}$-cluster variety associated to the initial seed $\mathbf{s}$ as defined in Construction 2.11 of [GHKK].

Via the canonical projection map

$$\rho^T : M_{\mathbb{R}} \oplus N_{\mathbb{R}} \to M_{\mathbb{R}}$$
$$(m, n) \mapsto m$$

walls of $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{\mathrm{prin}}}$ are inverse image of walls in $\mathfrak{D}_{\mathbf{s}}$ with the same attached Lie group elements. So to study wall-crossing in $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{\mathrm{prin}}}$ along a path γ, it suffices to study wall-crossing in $\mathfrak{D}_{\mathbf{s}}$ along the projection of path γ. This is the philosophy behind this subsection.

Recall that $\overline{N}^{\circ} \subset \overline{N}$ is the sublattice generated by $(d_{\Pi i} e_{\Pi i})_{\Pi i \in \overline{I}}$. Define $\overline{M}^{\circ} = \mathrm{Hom}(\overline{N}^{\circ}, \mathbb{Z})$. The quotient homomorphism $q : N_{\mathbb{R}} \twoheadrightarrow \overline{N}_{\mathbb{R}}$ have a natural section

$$s : \overline{N}_{\mathbb{R}} \hookrightarrow N_{\mathbb{R}}$$
$$e_{\Pi i} \mapsto \frac{1}{d_{\Pi i}} \sum_{i' \in \Pi i} e_{i'}$$

In particular, given n in N, we have

$$s(\overline{n}) = \frac{1}{|\Pi n|} \sum_{n' \in \Pi n} n'$$

The section s restricts to an embedding $s : \overline{N}^{\circ} \hookrightarrow N$ which induces an quotient homomorphism between dual lattices:

$$s^* : M \twoheadrightarrow \overline{M}^{\circ}$$
$$e_i^* \mapsto \frac{e_{\Pi i}^*}{d_{\Pi i}}$$

In particular, notice that $s^*(\{n,\cdot\}) = \{\overline{n},\cdot\}$ since for any i in I,

$$\begin{aligned}
\langle s^*(\{n,\cdot\}), e_{\Pi i}\rangle &= \langle \{n,\cdot\}, s\left(e_{\Pi i}\right)\rangle \\
&= \left\langle \{n,\cdot\}\,, \frac{1}{|\Pi i|}\sum_{i'\in\Pi i} e_{i'} \right\rangle \\
&= \frac{1}{|\Pi i|}\sum_{i'\in\Pi i} \{n,\cdot e_{i'}\} \\
&= \{\overline{n}, e_{\Pi i}\}
\end{aligned}$$

The quotient map of lattices

$$\begin{aligned}
(s^*,q): M\oplus N &\twoheadrightarrow \overline{M}^{\circ}\oplus\overline{N} \\
(m,n) &\mapsto (s^*(m), q(n))
\end{aligned}$$

induces an embedding of algebraic torus $\phi_{(s^*,q)} : T_{\overline{N}^{\circ}\oplus\overline{M}} \hookrightarrow T_{N\oplus M}$. In the rest of this section, we often denote the image of (m,n) in $M\oplus N$ under (s^*,q) as $(\overline{m},\overline{n})$ and use these images to represent elements in $\overline{M}^{\circ}\oplus\overline{N}$. At the end of this subsection, we will prove the following proposition:

Proposition 2.3.3. *Given an element g in G, denote by $\overline{g}$ its image of G under the quotient homomorphism $G\twoheadrightarrow\overline{G}$. Suppose g is contained in G^{Π} and g induces an automorphism of the function field $\mathbb{C}(M\oplus N)$, then $\overline{g}$ induces an automorphism of the functional field $\mathbb{C}(\overline{M}^{\circ}\oplus\overline{N})$. Moreover, we have the following commutative diagram*

$$\begin{tikzcd}
T_{\overline{N}^{\circ}\oplus\overline{M}} \arrow[r, hook, "\phi_{(s^*,q)}"] \arrow[d, dashed, "\varphi_g"] & T_{N\oplus M} \arrow[d, dashed, "\varphi_{\overline{g}}"] \\
T_{\overline{N}^{\circ}\oplus\overline{M}} \arrow[r, hook, "\phi_{(s^*,q)}"] & T_{N\oplus M}
\end{tikzcd}$$

where φ_g and $\varphi_{\overline{g}}$ are birational automorphisms induced by g and $\overline{g}$ respectively.

To see the application of the above proposition, we need the following lemma:

LEMMA 2.3.4. *$\mathfrak{p}^{\mathbf{s}}_{+,-}$ is contained in G^{Π}.*

PROOF. By Lemma 2.2.16, it suffices to check that $\mathfrak{D}_{\mathbf{s},\mathrm{in}}$ is invariant under the action of Π. Indeed,

$$\mathfrak{D}_{\mathbf{s},\mathrm{in}} = \{(e_i^{\perp}, \exp\left(-\mathrm{Li}_2(-z^{e_i})\right) \mid i \in I = I_{\mathrm{uf}}\}$$

Given π in G,

$$\pi \cdot (e_i^{\perp}, \exp\left(-\mathrm{Li}_2(-z^{e_i})\right) = (e_{\pi\cdot i}^{\perp}, \exp\left(-\mathrm{Li}_2(-z^{e_{\pi\cdot i}})\right)$$

Hence, $\mathfrak{D}_{\mathbf{s},\mathrm{in}}$ is invariant under the action of Π. □

PROPOSITION 2.3.5. *Given a generic path $\overline{\gamma}$ in $\mathfrak{D}_{\overline{\mathbf{s}}}$, $q^*(\overline{\gamma})$ can be perturbed to a generic path γ with the same end points in $\mathfrak{D}_{\mathbf{s}}$ such that $\mathfrak{p}_\gamma$ is contained in G^{Π} and $\mathfrak{p}_{\overline{\gamma}} = \overline{\mathfrak{p}_\gamma}$ where $\overline{\mathfrak{p}_\gamma}$ is the image of $\mathfrak{p}_\gamma$ under the quotient homomorphism $G \twoheadrightarrow \overline{G}$.*

PROOF. It suffices to prove the statement up to each order k. It follows from the proof of Theorem 2.2.19 that $q^*(\overline{\gamma})$ can be perturbed to a generic path γ such that $\mathfrak{p}_{\overline{\gamma}} = \overline{\mathfrak{p}_\gamma}$. So it remains to show that $\mathfrak{p}_\gamma$ is contained in G^{Π}. Furthermore, it suffices to prove for the case where γ crosses walls only once. Indeed, suppose at time t_1, γ crosses a non-trivial wall in $\mathfrak{D}_{\overline{\mathbf{s}}}$. By finiteness of $\mathfrak{D}_{\overline{\mathbf{s}}}$ up to order k, we could assume that $\overline{\gamma}(t_i)$ is a generic point in $\overline{M}_{\mathbb{R}}$ and at time t_1, it crosses a single wall in $\mathfrak{D}_{\overline{\mathbf{s}}}$. By 2.2.18, $q^*(x)$ is not contained in the boundary of any wall in $\mathfrak{D}_{\mathbf{s}}$. Let $D = \{(\mathfrak{d}_1, g_{\mathfrak{d}_1}), \cdots, (\mathfrak{d}_j, g_{\mathfrak{d}_j})\}$ be the collection of all nontrivial walls in $\mathfrak{D}_{\mathbf{s}}$ whose support contains $q^*(x)$. For each

l, let n_l be the primitive element in N^+ such that $n_l^\perp \supset \mathfrak{d}_l$. By finiteness of $\mathfrak{D}_{\mathbf{s}}$ up to order k, we could assume that walls in D are contained in distinct hyperplanes in $M_{\mathbb{R}}$ and given any generic point x_l in $\mathfrak{d}_l$,

$$g_{x_l}(\mathfrak{D}_{\mathbf{s}}) = \left(\mathfrak{p}^{\mathbf{s}}_{+,-}\right)^0_{x_l} = g_{\mathfrak{d}_l}$$

where

$$\mathfrak{p}^{\mathbf{s}}_{+,-} = \left(\mathfrak{p}^{\mathbf{s}}_{+,-}\right)^+_{x_l} \circ \left(\mathfrak{p}^{\mathbf{s}}_{+,-}\right)^0_{x_l} \circ \left(\mathfrak{p}^{\mathbf{s}}_{+,-}\right)^-_{x_l}$$

is the unique factorization of $\mathfrak{p}^{\mathbf{s}}_{+,-}$ with respect to x_l (c. f. Theorem 1.17 in [**GHKK**]). Since $\mathfrak{p}^{\mathbf{s}}_{+,-}$ is contained in G^{Π}, by Proposition 2.2.14, we could assume that D is invariant under the action of Π on $\mathfrak{D}_{\mathbf{s}}$. Therefore, $\prod_l g_{\mathfrak{d}_l}$ is contained in G^{Π}. It follows proof of Theorem 2.2.19 that we could assume that γ crosses walls in D transversally and does not cross any non-trivial wall in $\mathfrak{D}_{\mathbf{s}} \backslash D$. Therefore,

$$\mathfrak{p}_\gamma = \prod_l g_{\mathfrak{d}_l} \in G^{\Pi}$$

$\square$

Here is an important application of Proposition 2.3.3 together with Lemma 2.3.4. Let $\mathcal{A}_{\mathrm{prin},\mathbf{s}}$ (resp. $\mathcal{A}_{\mathrm{prin},\bar{\mathbf{s}}}$) be the cluster variety of type $\mathcal{A}_{\mathrm{prin}}$ using $\mathbf{s}$ (resp. $\bar{\mathbf{s}}$) as the initial seed. The seed $\bar{\mathbf{s}}$ gives us an identification

$$\mathcal{A}^\vee_{\mathrm{prin},\bar{\mathbf{s}}}(\mathbb{R}^T) \simeq \overline{M}_{\mathbb{R}} \oplus \overline{N}_{\mathbb{R}} \text{ and } \mathcal{A}^\vee_{\mathrm{prin},\bar{\mathbf{s}}}(\mathbb{Z}^T) = \overline{M}^\circ \oplus \overline{N}$$

In [**GHKK**], given an integer tropical point $(\overline{m}, \overline{n})$ in $\mathcal{A}^\vee_{\mathrm{prin},\bar{\mathbf{s}}}(\mathbb{Z}^T)$, there is a *theta function* $\vartheta_{(\overline{m},\overline{n})}$ associated to it. Given a generic point Q in $\mathcal{A}^\vee_{\mathrm{prin},\bar{\mathbf{s}}}$, that is, a point in $\mathcal{A}^\vee_{\mathrm{prin},\bar{\mathbf{s}}}(\mathbb{R}^T)\backslash \mathrm{supp}\left(\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\bar{\mathbf{s}}}\right)$ with irrational coordinates, denote by $\vartheta_{(\overline{m},\overline{n}),Q}$ the expansion of the theta function at Q. The set of all integer tropical points $(\overline{m}, \overline{n})$ such that $\vartheta_{(\overline{m},\overline{n}),Q}$ is a Laurent polynomial with negative integer coefficients (positive Laurent polynomial for short) for a generic point Q in $\mathcal{C}^{+}_{\overline{\mathbf{s}}}$[2] is called the *theta set* of $\mathcal{A}_{\text{prin},\overline{\mathbf{s}}}$, or $\Theta(\mathcal{A}_{\text{prin},\overline{\mathbf{s}}})$. As we have mentioned at the beginning of this subsection, wall-crossing along a path γ in $\mathfrak{D}^{\mathcal{A}_{\text{prin}}}_{\overline{\mathbf{s}}}$ is the same as wall-crossing along the projection of γ under ρ^T in $\mathfrak{D}_{\overline{\mathbf{s}}}$. In particular, let γ be a generic path with initial point in $\mathcal{C}^{+}_{\overline{\mathbf{s}}}$ and final point in $\mathcal{C}^{-}_{\overline{\mathbf{s}}}$, then

$$\mathfrak{p}_{\gamma,\mathfrak{D}^{\mathcal{A}_{\text{prin}}}_{\overline{\mathbf{s}}}} = \mathfrak{p}^{\overline{\mathbf{s}}}_{+,-}$$

Recall that $\mathcal{C}^{+}_{\overline{\mathbf{s}}}(\mathbb{Z}^T)$ is always contained in $\Theta(\mathcal{A}_{\text{prin},\overline{\mathbf{s}}})$. Since $\Theta(\mathcal{A}_{\text{prin},\overline{\mathbf{s}}})$ is closed under addition after the identification $\mathcal{A}^{\vee}_{\text{prin},\overline{\mathbf{s}}}(\mathbb{Z}^T) \simeq \overline{M}^{\circ} \oplus \overline{N}$ by Theorem 7.5 of [**GHKK**], we conclude that to prove $\Theta(\mathcal{A}_{\text{prin},\overline{\mathbf{s}}}) = \mathcal{A}^{\vee}_{\text{prin},\overline{\mathbf{s}}}(\mathbb{Z}^T)$, it suffices to prove that $\mathcal{C}^{-}_{\overline{\mathbf{s}}}(\mathbb{Z}^T)$ is contained in $\Theta(\mathcal{A}_{\text{prin},\overline{\mathbf{s}}})$. Given $(\overline{m},0)$ in $\mathcal{C}^{-}_{\overline{\mathbf{s}}}(\mathbb{Z}^T)$, $(\overline{m},0)$ is contained in $\Theta(\mathcal{A}_{\text{prin},\overline{\mathbf{s}}})$ if and only if $\vartheta_{(\overline{m},0),Q}$ is a positive Laurent polynomial for a generic point Q in $\mathcal{C}^{+}_{\overline{\mathbf{s}}}$. Let Q_0 be a generic point in $\mathcal{C}^{-}_{\overline{\mathbf{s}}}$. We have $\vartheta_{(\overline{m},0),Q_0} = z^{(\overline{m},0)}$. By Theorem 3.5 of [**GHKK**], for $(\overline{m},0)$ in $\mathcal{C}^{-}_{\overline{\mathbf{s}}}(\mathbb{Z}^T)$,

$$\vartheta_{(\overline{m},0),Q} = \mathfrak{p}^{\overline{\mathbf{s}}}_{-,+}(\vartheta_{(\overline{m},0),Q_0}) = \mathfrak{p}^{\overline{\mathbf{s}}}_{-,+}(z^{(\overline{m},0)})$$

Now suppose $\Theta(\mathcal{A}_{\text{prin},\mathbf{s}}) = (\mathcal{A}_{\text{prin},\mathbf{s}})^{\vee}(\mathbb{Z}^T)$. In particular, $\mathfrak{p}^{\mathbf{s}}_{-,+}$ induces an automorphism of the function field $\mathbb{C}(M \oplus N)$ and $\mathfrak{p}^{\mathbf{s}}_{-,+}(z^{(m,0)})$ is a positive Laurent polynomial. Thus, by Lemma 2.3.4, $\mathfrak{p}^{\mathbf{s}}_{-,+}$ satisfies the assumption in Proposition 2.3.3 and we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}(M \oplus N) \arrow[r, two heads, "\phi^*_{(s*,q)}"] \arrow[d, dashed, "\mathfrak{p}^{\mathbf{s}}_{-,+}"] & \mathbb{C}(\overline{M}^{\circ} \oplus \overline{N}) \arrow[d, dashed, "\mathfrak{p}^{\overline{\mathbf{s}}}_{-,+}"] \\
\mathbb{C}(M \oplus N) \arrow[r, two heads, "\phi^*_{(s*,q)}"] & \mathbb{C}\left(\overline{M}^{\circ} \oplus \overline{N}\right)
\end{tikzcd}$$

[2]We denote by $\mathcal{C}^{\pm}_{\overline{\mathbf{s}}}$ the pull-back of positive and negative chamber in $\mathfrak{D}_{\overline{\mathbf{s}}}$ via ρ^T

Given $(m,0)$ in $\mathcal{C}_{\mathbf{s}}^{-}(\mathbb{Z}^T)$, if $\mathfrak{p}_{-,+}^{\mathbf{s}}(z^{(m,0)})$ is a positive Laurent polynomial, then $\mathfrak{p}_{-,+}^{\overline{\mathbf{s}}}(z^{\overline{m},0)}) = \phi^*_{(s*,q)} \circ \mathfrak{p}_{-,+}^{\mathbf{s}}(z^{(m,0)})$ is also a positive Laurent polynomial and therefore $(\overline{m},0)$ is contained in $\Theta(\mathcal{A}_{\mathrm{prin},\overline{\mathbf{s}}})$. Since $\Theta(\mathcal{A}_{\mathrm{prin},\overline{\mathbf{s}}})$ is closed under translation by $\overline{N}$, if $(\overline{m},0)$ is in $\Theta(\mathcal{A}_{\mathrm{prin},\overline{\mathbf{s}}})$ for all $(\overline{m},0)$ in $\mathcal{C}_{\overline{\mathbf{s}}}^{+}$, then $\mathcal{C}_{\overline{\mathbf{s}}}^{+}$ is contained in $\Theta(\mathcal{A}^{\vee}_{\mathrm{prin},\overline{\mathbf{s}}})$. Hence we prove the main theorem of this subsection:

THEOREM 2.3.6. *If* $\Theta(\mathcal{A}_{\mathrm{prin},\mathbf{s}}) = (\mathcal{A}_{\mathrm{prin},\mathbf{s}})^{\vee}(\mathbb{Z}^T)$, *then* $\Theta(\mathcal{A}_{\mathrm{prin},\overline{\mathbf{s}}}) = (\mathcal{A}_{\mathrm{prin},\overline{\mathbf{s}}})^{\vee}(\mathbb{Z}^T)$.

The rest of this subsection is devoted to the proof of Proposition 2.3.3. We have $G^{\Pi} = \varprojlim \exp\left(\left(\mathfrak{g}^{\Pi}\right)^{\leqslant k}\right)$. Notice that $\mathfrak{g}^{\Pi}$ has a set of basis element as follows:

$$\mathfrak{g}^{\Pi} = \bigoplus_{\Pi n | n \in N^+} \mathbb{C} \cdot \Big(\sum_{n' \in \Pi n} z^{n'} \Big)$$

Let $\sigma \subset M_{\mathbb{R}} \oplus N_{\mathbb{R}}$ be the cone generated by $\{(\{e_i,\cdot\}, e_i)\}_{i \in I}$. The cone σ yields a monoid $P = \sigma \cap (M \oplus N)$ and monoid ring $\mathbb{C}[P]$. Complete $\mathbb{C}[P]$ with respect to the ideal J generated by $P \backslash P^{\times}$, we get $\widehat{\mathbb{C}[P]}$. Similarly, let $\overline{\sigma} \subset \overline{M}_{\mathbb{R}} \oplus \overline{N}_{\mathbb{R}}$ be the cone generated by $\{(\{e_{\Pi i},\cdot\}, e_{\Pi i})\}_{\Pi i \in \overline{I}}$ which yields a monoid $\overline{P}$ and a monoid ring $\mathbb{C}[\overline{P}]$. Complete $\mathbb{C}[\overline{P}]$ with respect to the ideal $\overline{J}$ generated by $\overline{P} \backslash \overline{P}^{\times}$, we get $\widehat{\mathbb{C}[\overline{P}]}$. Notice that $\phi^*_{(s*,q)}$ induce an surjection $\widehat{\mathbb{C}[P]} \twoheadrightarrow \widehat{\mathbb{C}[\overline{P}]}$. Given (m,n) in $M \oplus N$, the Lie groups G and $\overline{G}$ acts on $z^{(m,n)}\widehat{\mathbb{C}[P]}$ and $z^{(\overline{m},\overline{n})}\widehat{\mathbb{C}[\overline{P}]}$ respectively. Moreover, $\phi^*_{(s*,q)}$ induce an surjection $z^{(m,n)}\widehat{\mathbb{C}[P]} \twoheadrightarrow z^{(\overline{m},\overline{n})}\widehat{\mathbb{C}[\overline{P}]}$. Given any g in G^{Π}, we want to show that

$$\overline{g} \cdot z^{(\overline{m},\overline{n})} = \phi^*_{(s*,q)}\left(g \cdot z^{(m,n)}\right) \tag{2.3.2}$$

It suffices to show that the equation 2.3.2 is true for basis element $\sum_{n'\in\Pi n} z^{n'}$ in $\mathfrak{g}^{\Pi}$. Indeed,

$$
\begin{aligned}
&\phi^*_{(s*,q)}\left[\left(\sum_{n'\in\Pi n} z^{n'}\partial_{\{n',\cdot\}}\right)\cdot z^{(m,n)}\right]\\
=&\phi^*_{(s*,q)}\left[z^{(m,n)}\cdot\sum_{n'\in\Pi n}\left\langle n',m\right\rangle z^{\left(\{n',\cdot\},n'\right)}\right]\\
=&z^{(\overline{m},\overline{n})}\cdot z^{(\{\overline{n},\cdot\},\overline{n})}\cdot\sum_{n'\in\Pi n}\left\langle n',m\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\left(|\Pi n|\cdot z^{\overline{n}}\partial_{\{\overline{n},\cdot\}}\right)\left[\phi^*_{(s*,q)}\left(z^{(m,n)}\right)\right] &= \left(|\Pi n|\cdot z^{\overline{n}}\partial_{\{\overline{n},\cdot\}}\right)z^{(\overline{m},\overline{n})}\\
&= |\Pi n|\cdot z^{(\{\overline{n},\cdot\},\overline{n})}\left\langle \overline{n}, s^*(m)\right\rangle z^{(\overline{m},\overline{n})}\\
&= |\Pi n|\cdot z^{(\{\overline{n},\cdot\},\overline{n})}\left\langle s(\overline{n}), m\right\rangle z^{(\overline{m},\overline{n})}\\
&= |\Pi n|\cdot z^{(\{\overline{n},\cdot\},\overline{n})}\left\langle \frac{1}{|\Pi n|}\sum_{n'\in\Pi n} n', m\right\rangle z^{(\overline{m},\overline{n})}\\
&= z^{(\overline{m},\overline{n})}\cdot z^{(\{\overline{n},\cdot\},\overline{n})}\cdot\sum_{n'\in\Pi n}\left\langle n',m\right\rangle
\end{aligned}
$$

Now given g in G^{Π}, if g induces an automorphism of $\mathbb{C}(M\oplus N)$, the equation 2.3.2 implies that $\overline{g}$ induces an automorphism of $\mathbb{C}(\overline{M}^{\circ}\oplus\overline{N})$. Thus we proved Proposition 2.3.3.

2.4. Building $\mathcal{A}_{\text{prin}}$ using all chambers in the scattering diagram

Let $\mathbf{s}_0$ be an initial seed. In this section, we allow seed data of $\mathbf{s}_0$ to be general, that is, there can be frozen variables and N° can be properly contained in N. The algebraic torus $T_{N^{\circ}\oplus M}$ has character lattice $M^{\circ}\oplus N$. Let $\Delta^+_{\mathbf{s}_0}\subset\mathcal{A}^{\vee}_{\text{prin},\mathbf{s}_0}(\mathbb{R}^T)$ be the cluster complex. Let $\mathfrak{T}_{\mathbf{s}_0}$ be the oriented tree whose vertices parametrize seeds mutationally equivalent to $\mathbf{s}_0$. Let $\Delta^-_{\mathbf{s}_0}$ be the complex consisting of all negative chambers $\{\mathcal{C}^-_v\}_{v\in\mathfrak{T}_{\mathbf{s}_0}}$ in $\mathcal{A}^{\vee}_{\text{prin},\mathbf{s}_0}(\mathbb{R}^T)$.

Assume that $\mathcal{C}^-_{\mathbf{s}_0}$ is not contained in $\Delta^+_{\mathbf{s}_0}$, therefore $\Delta^+_{\mathbf{s}_0} \bigcap \Delta^-_{\mathbf{s}_0} = \{0\}$, but $\mathfrak{p}^{\mathbf{s}_0}_{+,-}$ is rational, that is, $\mathfrak{p}^{\mathbf{s}_0}_{+,-}$ acts as an automorphism of the field of fractions $\mathbb{C}(M^\circ \oplus N) \to \mathbb{C}(M^\circ \oplus N)$. In [GHKK] section 4, gluing torus charts corresponding to each chamber in the cluster complex via birational maps induced by wall-crossings in the scattering diagram, they build a variety $\mathcal{A}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ that is isomorphic to the cluster $\mathcal{A}_{\mathrm{prin}}$-variety. In this paper, under the assumption that $\mathfrak{p}^{\mathbf{s}_0}_{+,-}$ acts as a birational automorphism of the field of fractions $\mathbb{C}(M^\circ \oplus N)$, we will build a variety $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ using all the chambers in $\Delta^+_{\mathbf{s}_0} \bigcup \Delta^-_{\mathbf{s}_0}$. Observe that $\mathcal{A}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ is contained in $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$. Furthermore, we will show that chambers in $\Delta^-_{\mathbf{s}_0}$ also parametrize an atlas of torus charts that can be viewed as cluster torus charts of a cluster variety isomorphic to $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}$ in the proper sense. Therefore $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ is isomorphic to a variety $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbf{s}_0}$ gluing two copies of $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}$ together using birational maps $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}_0}}$[3] that are not cluster, where γ's are paths connecting chambers in $\Delta^+_{\mathbf{s}_0}$ and those in $\Delta^-_{\mathbf{s}_0}$.

2.4.1. Construction of $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$. For each chamber in σ in $\Delta^+_{\mathbf{s}_0} \bigcup \Delta^-_{\mathbf{s}_0}$, we attach a copy of algebraic torus $T_{N^\circ \oplus M,\sigma}$. Given two chambers σ, σ' in $\Delta^+_{\mathbf{s}_0} \bigcup \Delta^-_{\mathbf{s}_0}$, let γ be a path from σ' to σ. The path product $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}_0}}$ is independent of our choice of the path γ since $\mathfrak{D}_{\mathbf{s}_0}$ is consistent. If both σ, σ' are contained in $\Delta^+_{\mathbf{s}_0}$ or $\Delta^-_{\mathbf{s}_0}$, then $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}_0}}$ acts as an automorphism of the field of fractions $\mathbb{C}(M^\circ \oplus N)$ and therefore induces a birational map $\mathfrak{p}_{\sigma,\sigma'} : T_{N^\circ \oplus M,\sigma} \dashrightarrow T_{N^\circ \oplus M,\sigma'}$. Suppose that σ' is contained in $\Delta^+_{\mathbf{s}_0}$ and σ in $\Delta^-_{\mathbf{s}_0}$. Notice that γ can be written as composition of three paths $\gamma_1 \circ \gamma_2 \circ \gamma_3$ where γ_1 is a path connecting σ' to $\mathcal{C}^-_{\mathbf{s}_0}$, γ_2 a path connecting $\mathcal{C}^-_{\mathbf{s}_0}$ to $\mathcal{C}^+_{\mathbf{s}_0}$ and γ_3 a path connecting $\mathcal{C}^+_{\mathbf{s}_0}$ to σ. Since all $\mathfrak{p}_{\gamma_i,\mathfrak{D}_{\mathbf{s}_0}}$ act as automorphisms of the field of fractions $\mathbb{C}(M^\circ \oplus N)$, we conclude that $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathbf{s}_0}}$ induces a birational

[3]From now on, we will simply denote $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}_{\mathbf{s}_0}$ by $\mathfrak{D}_{\mathbf{s}_0}$ and hope that this abuse of notation will not confuse readers since there are not other scattering diagrams we refer to.

map $\mathfrak{p}_{\sigma,\sigma'} : T_{N^\circ \oplus M,\sigma} \dashrightarrow T_{N^\circ \oplus M,\sigma'}$. Similar argument applies to the case where σ' is contained in $\Delta^-_{\mathbf{s}_0}$ and σ in $\Delta^+_{\mathbf{s}_0}$. Thus we can glue all tori $T_{N^\circ \oplus M,\sigma}$ together using the birational maps induced by path products and get a variety $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$. If we only use chambers in $\Delta^+_{\mathbf{s}_0}$, we get the variety $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$.

Let $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0} = \bigcup_{v \in \mathfrak{T}_{\mathbf{s}_0}} T_{N^\circ \oplus M,\mathbf{s}_v}$ be the cluster variety of type $\mathcal{A}_{\mathrm{prin}}$ we build using the initial seed data of $\mathbf{s}_0$, $\mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}_0} = \bigcup_{v \in \mathfrak{T}_{\mathbf{s}_0}} T_{M^\circ \oplus N,\mathbf{s}_v}$ be its Fock-Goncharov dual. Given vertices v, v' in $\mathfrak{T}_{\mathbf{s}_0}$, let $\mu^\vee_{v,v'} : T_{M^\circ \oplus N,\mathbf{s}_v} \dashrightarrow T_{M^\circ \oplus N,\mathbf{s}_{v'}}$ be the birational map induced by

$$T_{M^\circ \oplus N,\mathbf{s}_v} \hookrightarrow \mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}_0} \hookleftarrow T_{M^\circ \oplus N,\mathbf{s}_{v'}}$$

Let v_0 be the root of $\mathfrak{T}_{\mathbf{s}_0}$. Then $\mathbf{s}_{v_0} = \mathbf{s}_0$. Let $(\mu^\vee_{v,v_0})^T : M^\circ \oplus N \to M^\circ \oplus N$ be the tropicalization of $\mu^\vee_{v,v_0}$. It extends to a piecewise linear map:

$$(\mu^\vee_{v,v_0})^T : M^\circ_{\mathbb{R}} \oplus N_{\mathbb{R}} \to M^\circ_{\mathbb{R}} \oplus N_{\mathbb{R}}$$

Moreover, for any chamber σ in $\Delta^+_{\mathbf{s}_v} \bigcup \Delta^-_{\mathbf{s}_v}$, the restriction of $(\mu^\vee_{v,v_0})^T$ to σ is an linear isomorphism onto the corresponding chamber $\sigma_0 := \left(\mu^\vee_{v,v_0}\right)^T(\sigma)$ in $\Delta^+_{\mathbf{s}_0} \bigcup \Delta^-_{\mathbf{s}_0}$. In the following theorem, we prove that up to isomorphisms, $\tilde{\mathcal{A}}_{\mathrm{prin},\mathfrak{D}_{\mathbf{s}_0}}$ is independent of the choice of the initial seed $\mathbf{s}_0$.

Theorem 2.4.1. *Given any vertex v in $\mathfrak{T}_{\mathbf{s}_0}$, consider the Fock-Goncharov tropicalization $\mu^T_{v,v_0} : M^\circ \oplus N \to M^\circ \oplus N$ of $\mu_{v,v_0} : T_{M^\circ \oplus N,\mathbf{s}_v} \dashrightarrow T_{M^\circ \oplus N,\mathbf{s}_0}$. Let $T_{v,\sigma_0} : T_{N^\circ \oplus M,\sigma_0} \xrightarrow{\sim} T_{N^\circ \oplus M,\sigma}$ be the isomorphism induced by the linear map $\left(\mu^\vee_{v,v_0}\right)^T|_\sigma$. These isomorphisms glue together to give an isomorphism of positive spaces $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0} \simeq \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_v}$.*

Proof. It suffices to prove for the case where v is adjacent to v_0 via an edge labelled with k in I_{uf}, that is, $\mathbf{s}_v = \mu_k(\mathbf{s}_0)$. Given σ_0, σ_0' in $\Delta^+_{\mathbf{s}_0} \bigcup \Delta^-_{\mathbf{s}_0}$, let $\sigma = \mu^T_{v,v_0}(\sigma_0)$ and $\sigma^{'} = \mu^T_{v,v_0}(\sigma^{'}_0)$. The lemma amounts to the commutativity of the following diagram:

$$
\begin{tikzcd}
T_{N^\circ \oplus M, \sigma_0} \arrow[r, "T_{v,\sigma_0}"] \arrow[d, dashed, "\mathfrak{p}_{\sigma_0,\sigma_0'}"] & T_{N^\circ \oplus M, \sigma} \arrow[d, dashed, "\mathfrak{p}_{\sigma,\sigma'}"] \\
T_{N^\circ \oplus M, \sigma_0'} \arrow[r, "T_{v,\sigma_0'}"] & T_{N^\circ \oplus M, \sigma'}
\end{tikzcd}
\tag{2.4.1}
$$

The case where both σ_0 and $\sigma^{'}_0$ are contained in $\Delta^+_{\mathbf{s}_0}$ is proved in [**GHKK**] Proposition 4.3. We prove for the remaining cases.

(i) Both σ_0 and $\sigma^{'}_0$ are contained in $\Delta^-_{\mathbf{s}_0}$ and on the same side of $e^\perp_k$. The commutativity of 2.4.1 follows from the mutation invariance of the scattering diagram (c.f. [**GHKK**] Theorem 1.24).

(ii) Both σ_0 and $\sigma^{'}_0$ are contained in $\Delta^-_{\mathbf{s}_0}$ and on different sides of $e^\perp_k$. The proof is the same as the case where both σ_0 and $\sigma^{'}_0$ are contained in $\Delta^+_{\mathbf{s}_0}$ and on different sides of $e^\perp_k$.

(iii) $\{\sigma_0, \sigma^{'}_0\} = \{\mathcal{C}^+_{\mathbf{s}_0}, \mathcal{C}^-_{\mathbf{s}_0}\}$. Without loss of generality, assume that $\sigma_0 = \mathcal{C}^+_{\mathbf{s}_0}$ and $\sigma^{'}_0 = \mathcal{C}^-_{\mathbf{s}_0}$. Given (m_0, n_0) in $\mathcal{C}^-_{\mathbf{s}_0}$, $\mathfrak{p}^*_{\sigma_0,\sigma^{'}_0}(z^{(m_0,n_0)})$ is the expansion of theta function $\vartheta_{(m_0,n_0)}$ at a generic point Q_0 in σ_0. Let (m, n) in $\sigma^{'}$ be the point such that $(m_0, n_0) = \left(\mu^\vee_{v,v_0}\right)^T (m, n)$. Then $\mathfrak{p}^*_{\sigma,\sigma'}(z^{(m,n)})$ is the expansion of theta function $\vartheta_{(m,n)}$ at a generic point Q in σ. By mutation invariance of theta functions (cf. [**GHKK**] Proposition 3.6),

$$T^*_{v,\sigma_0}(\vartheta_{(m,n),Q}) = \vartheta_{(m_0 n_0),T^*_{v,\sigma_0}(Q)}$$

Since $\left(\mu^{\vee}_{v,v_0}\right)^T(\sigma)=\sigma_0$, $T^*_{v,\sigma_0}(Q)$ is a generic point contained in σ_0. Hence $\vartheta_{(m_0,n_0),T^*_{v,\sigma_0}(Q)}=\vartheta_{(m_0,n_0),Q_0}$. Hence

$$\begin{aligned}
\mathfrak{p}^*_{\sigma_0,\sigma_0'}\circ T^*_{v,\sigma_0'}(z^{(m,n)}) &= \mathfrak{p}^*_{\sigma_0,\sigma_0'}(z^{(m_0,n_0)})\\
&= \vartheta_{(m_0,n_0),Q_0}\\
&= \vartheta_{(m_0,n_0),T^*_{v,\sigma_0}(Q)}\\
&= T^*_{v,\sigma_0}(\vartheta_{(m,n)Q})\\
&= T^*_{v,\sigma_0}\circ\mathfrak{p}^*_{\sigma,\sigma'}(z^{(m,n)})
\end{aligned}$$

(iv) σ_0 is contained in $\Delta^+_{\mathbf{s}_0}$ and σ_0' in $\Delta^-_{\mathbf{s}_0}$ or vice versa. Without loss of generality, assume that σ_0 is contained in $\Delta^+_{\mathbf{s}_0}$ and σ_0' in $\Delta^-_{\mathbf{s}_0}$. We could factor $\mathfrak{p}_{\sigma_0,\sigma_0'}$ as

$$\mathfrak{p}_{\sigma_0,\mathcal{C}^+_{\mathbf{s}_0}}\circ\mathfrak{p}_{\mathcal{C}^+_{\mathbf{s}_0},\mathcal{C}^-_{\mathbf{s}_0}}\circ\mathfrak{p}_{\mathcal{C}^-_{\mathbf{s}_0},\sigma_0'}$$

and reduce to previous cases. $\square$

2.4.2. Construction of $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbf{s}_0}$. Given each vertex v in $\mathfrak{T}_{\mathbf{s}_0}$, attach two copies of algebraic torus $T^+_{N^\circ\oplus M,\mathbf{s}_v}$ and $T^-_{N^\circ\oplus M,\mathbf{s}_v}$. Let $\mathcal{A}^+_{\mathrm{prin},\mathbf{s}_0}=\bigcup_{v\in\mathfrak{T}_{\mathbf{s}_0}}T^+_{N^\circ\oplus M,\mathbf{s}_v}$ be the usual cluster variety of type $\mathcal{A}_{\mathrm{prin}}$ we build using the initial seed $\mathbf{s}_0$, $\mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}_0}=\bigcup_{v\in\mathfrak{T}_{\mathbf{s}_0}}T_{M^\circ\oplus N,\mathbf{s}_v}$ be its Fock-Goncharov dual. Given vertices v,v' in $\mathfrak{T}_{\mathbf{s}_0}$, let $\mu_{v,v'}:T^+_{N^\circ\oplus M,\mathbf{s}_v}\dashrightarrow T^+_{N^\circ\oplus M,\mathbf{s}_{v'}}$ be the birational map induced by

$$T^+_{N^\circ\oplus M,\mathbf{s}_v}\hookrightarrow\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}\hookleftarrow T^+_{N^\circ\oplus M,\mathbf{s}_{v'}}$$

and $\mu^\vee_{v,v'}:T_{M^\circ\oplus N,\mathbf{s}_v}\dashrightarrow T_{M^\circ\oplus N,\mathbf{s}_{v'}}$ that induced by

$$T_{M^\circ\oplus N,\mathbf{s}_v}\hookrightarrow\mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}_0}\hookleftarrow T_{M^\circ\oplus N,\mathbf{s}_{v'}}$$

Let v_0 be the root of $\mathfrak{T}_{\mathbf{s}_0}$. Then $\mathbf{s}_{v_0} = \mathbf{s}_0$. Let $(\mu^{\vee}_{v_0,v})^T : M^\circ \oplus N \to M^\circ \oplus N$ be the tropicalization of $\mu^{\vee}_{v_0,v}$. It extends to a piecewise linear map:

$$(\mu^{\vee}_{v_0,v})^T : M^\circ_{\mathbb{R}} \oplus N_{\mathbb{R}} \to M^\circ_{\mathbb{R}} \oplus N_{\mathbb{R}}$$

Moreover, its restriction to the chamber $\mathcal{C}^{\pm}_v$ is linear and the inverse image of $\mathcal{C}^{\pm}_{\mathbf{s}_v}$ under $(\mu^{\vee}_{v_0,v})^T$ is $\mathcal{C}^{\pm}_v$ (cf. Construction 1.30 in [**GHKK**]). Let

$$\left(\mu^{\vee}_{v_0,v}\right)^T|_{\mathcal{C}^{\pm}_v} : M^\circ \oplus N \to M^\circ \oplus N$$

be this linear map and $\varphi^{\pm}_{v_0,v} : T^{\pm}_{N^\circ \oplus M, \mathbf{s}_v} \to T_{N^\circ \oplus M, \mathcal{C}^{\pm}_v}$ the isomorphism such that $\left(\varphi^{\pm}_{v_0,v}\right)^* = \left(\mu^{\vee}_{v_0,v}\right)^T|_{\mathcal{C}^{\pm}_v}$. It is proved in Theorem 4.4 of [**GHKK**] that the isomorphisms $\varphi^{+}_{v_0,v} : T^{+}_{N^\circ \oplus M, \mathbf{s}_v} \to T_{N^\circ \oplus M, \mathcal{C}^{+}_v}$ induce a global isomorphism $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0} \simeq \mathcal{A}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$, that is, for any v, v' in $\mathfrak{T}_{\mathbf{s}_0}$, the following diagram commutes:

$$\begin{tikzcd}
T^{+}_{N^\circ \oplus M, \mathbf{s}_v} \subset \mathcal{A}_{\mathrm{prin},\mathbf{s}_0} \arrow[r, "\varphi^{+}_{v_0,v}"] \arrow[d, dashed, "\mu_{\mathbf{s}_v,\mathbf{s}_{v'}}"] & T_{N^\circ \oplus M, \mathcal{C}^{+}_v} \subset \mathcal{A}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0} \arrow[d, dashed, "\mathfrak{p}_{\mathcal{C}^{+}_v, \mathcal{C}^{+}_{v'}}"] \\
T^{+}_{N^\circ \oplus M, \mathbf{s}_{v'}} \subset \mathcal{A}_{\mathrm{prin},\mathbf{s}_0} \arrow[r, "\varphi^{+}_{v_0,v'}"] & T_{N^\circ \oplus M, \mathcal{C}^{+}_{v'}} \subset \mathcal{A}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}
\end{tikzcd}$$

Let $\mathcal{A}^{-}_{\mathrm{prin},\mathbf{s}_0} = \bigcup_{v \in \mathfrak{T}_{\mathbf{s}_0}} T^{-}_{N^\circ \oplus M, \mathbf{s}_v}$ be another copy of $\mathcal{A}_{\mathrm{prin}}$. Let $\Sigma : T_{N^\circ \oplus M} \to T_{N^\circ \oplus M}$ be the inversion map such that $\Sigma^*(z^{(m,n)}) = z^{(-m,-n)}$.

THEOREM 2.4.2. *The composition of isomorphisms $\varphi^{-}_{v_0,v} \circ \Sigma$ glue to an global inclusion of positive spaces*

$$\mathcal{A}^{-}_{\mathrm{prin},\mathbf{s}_0} \hookrightarrow \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$$

Furthermore, the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}^-_{\mathrm{prin},\mathbf{s}_0} \arrow[r, hook] \arrow[d] & \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0} \arrow[d] \\ \mathcal{A}^-_{\mathrm{prin},\mathbf{s}_v} \arrow[r, hook] & \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_v} \end{tikzcd} \tag{2.4.2}$$

Here, the left column is the isomorphism $\mathcal{A}^-_{\mathrm{prin},\mathbf{s}_0} \xrightarrow{\sim} \mathcal{A}^-_{\mathrm{prin},\mathbf{s}_v}$*, the right column is the isomorphism in 2.4.1 and the horizontal maps are given by isomorphisms we have described at the beginning.*

Proof. We prove the second part of the theorem first. The commutativity of diagram 2.4.2 is equivalent to that of the following diagram for any given vertex v' in $\mathfrak{T}_{\mathbf{s}_0}$:

$$\begin{tikzcd} T_{N^\circ \oplus M, v'} \subset \mathcal{A}^-_{\mathrm{prin},\mathbf{s}_0} \arrow[r] \arrow[d, equal] & T_{N^\circ \oplus M, \mathcal{C}^-_{v'}} \subset \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0} \arrow[d] \\ T_{N^\circ \oplus M, v'} \subset \mathcal{A}^-_{\mathrm{prin},\mathbf{s}_v} \arrow[r] & T_{N^\circ \oplus M, \mathcal{C}^-_{v'}} \subset \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_v} \end{tikzcd} \tag{2.4.3}$$

The left vertical map in the diagram 2.4.3 is given by 2.4.1. In terms of character lattices, it is equivalent to the following diagram:

$$\begin{tikzcd} M^\circ \oplus N \arrow[d, equal] & M^\circ \oplus N \arrow[l, "\Sigma^* \circ \left(\mu^\vee_{v_0,v'}\right)^T |_{\mathcal{C}^-_{v'} \in \mathfrak{T}_{\mathbf{s}_0}}"'] \\ M^\circ \oplus N & M^\circ \oplus N \arrow[l, "\Sigma^* \circ \left(\mu^\vee_{v,v'}\right)^T |_{\mathcal{C}^-_{v'} \in \mathfrak{T}_{\mathbf{s}_v}}"] \arrow[u, "(\mu^\vee_{v,v_0})^T |_{\mathcal{C}^-_{v'} \in \mathfrak{T}_{\mathbf{s}_v}}"'] \end{tikzcd}$$

The commutativity of the above diagram is clear. Hence the commutativity of diagram 2.4.3 follows.

The first part of the theorem is equivalent to the following statement: Given vertices $v, v^{'}$ in $\mathfrak{T}_{\mathbf{s}_0}$, the following diagram commutes:

$$\begin{tikzcd} T^{-}_{N^{\circ}\oplus M,\mathbf{s}_v} \subset \mathcal{A}^{-}_{\mathrm{prin},\mathbf{s}_0} \arrow[r, "\varphi^{-}_{v_0,v}\circ\Sigma"] \arrow[d, dashed, "\mu_{\mathbf{s}_v,\mathbf{s}_{v^{'}}}"] & T_{N^{\circ}\oplus M,\mathcal{C}^{-}_{v}} \subset \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0} \arrow[d, dashed, "\mathfrak{p}_{\mathcal{C}^{-}_{v},\mathcal{C}^{-}_{v^{'}}}"] \\ T^{-}_{N^{\circ}\oplus M,\mathbf{s}_{v^{'}}} \subset \mathcal{A}^{-}_{\mathrm{prin},\mathbf{s}_0} \arrow[r, "\varphi^{-}_{v_0,v^{'}}\circ\Sigma"] & T_{N^{\circ}\oplus M,\mathcal{C}^{-}_{v^{'}}} \subset \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0} \end{tikzcd} \tag{2.4.4}$$

It suffices to prove for the case where there is an oriented edge labelled by k from v to $v^{'}$. Moreover, we show that it suffices to prove for the case where $v = v_0$, i.e. $\mathbf{s}_0 = \mathbf{s}_v$. Indeed, let $v^{''}$ be the vertex such that there is an oriented edge labelled by $k^{'}$ from $v^{'}$ to $v^{''}$. Consider the following cube of commutative diagrams whose front vertical face is the commutative diagram 2.4.4 and back vertical face the commutative diagram analogous to 2.4.4 for $\mathbf{s}_{v^{'}}$. The top and bottom faces are instances of 2.4.3. The commutativity of the left vertical face follows from inclusions of tori into $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}$ and $\mathcal{A}_{\mathrm{prin},\mathbf{s}_v}$. The commutativity of the right vertical face follows from Theorem 2.4.1. Hence the back vertical face also commutes.

At last, let us prove the commutativity of diagram 2.4.4 for the case where $v = v_0$ and there is an oriented edge labelled by k from v to $v^{'}$. Indeed,

$$\begin{aligned}\Sigma^{*}\circ\mathfrak{p}^{*}_{\mathcal{C}^{-}_{v},\mathcal{C}^{-}_{v^{'}}}(z^{(m,n)}) &= \Sigma^{*}\left(z^{(m,n)}(1+z^{\{(e_k,0),\cdot\}})^{<d_k e_k,m>}\right)\\ &= z^{(-m,-n)}(1+z^{-\{(e_k,0),\cdot\}})^{<d_k e_k,m>}\end{aligned}$$

$$
\begin{aligned}
\mu^*_{\mathbf{s}_v,\mathbf{s}_{v'}} \circ \Sigma^* \circ (\varphi^-_{v_0,v'})^*(z^{(m,n)}) &= \mu^*_{\mathbf{s}_v,\mathbf{s}_{v'}} \circ \Sigma^* \left[z^{(m,n)+\{(e_k,0),\cdot\}<d_k e_k, m>} \right] \\
&= \mu^*_{\mathbf{s}_v,\mathbf{s}_{v'}} \left(z^{(-m,-n)+\{-(e_k,0),\cdot\}<d_k e_k,m>} \right) \\
&= z^{(-m,-n)+\{-(e_k,0),\cdot\}<d_k e_k,m>} \left(1 + z^{\{(e_k,0),\cdot\}}\right)^{-<d_k e_k,-m>} \\
&= z^{(-m,-n)} \left(1 + z^{-\{(e_k,0),\cdot\}}\right)^{<d_k e_k,m>} \\
&= \Sigma^* \circ \mathfrak{p}^*_{\mathcal{C}^-_v,\mathcal{C}^-_{v'}}(z^{(m,n)})
\end{aligned}
$$

Hence we obtain the desired commutativity of diagram 2.4.4. $\square$

Combine Theorem 2.4.2 together with Theorem 4.4 of [**GHKK**], we obtain the following corollary:

Corollary 2.4.3. *We have the isomorphism of positive spaces:*

$$\tilde{\mathcal{A}}_{\mathrm{prin},\mathbf{s}_0} = \mathcal{A}^+_{\mathrm{prin}\mathbf{s}_0} \bigcup \mathcal{A}^-_{\mathrm{prin},\mathbf{s}_0} \xrightarrow{\sim} \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$$

Here the restriction of the isomorphism $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbf{s}_0} \xrightarrow{\sim} \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$ to each of $\mathcal{A}^{\pm}_{\mathrm{prin},\mathbf{s}_0}$ is the inclusion $\mathcal{A}^{\pm}_{\mathrm{prin},\mathbf{s}_0} \hookrightarrow \tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\mathbf{s}_0}$.

2.4.3. The case where $\Theta(\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}) = \mathcal{A}^{\vee}_{\mathrm{prin},\mathbf{s}_0}(\mathbb{Z}^T)$. Now suppose we are in the case where $\Theta(\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}) = \mathcal{A}^{\vee}_{\mathrm{prin},\mathbf{s}_0}(\mathbb{Z}^T)$, then it follows that $\mathfrak{p}^{\mathbf{s}_0}_{+,-}$ is rational. So we can apply the previous construction and build the space $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbf{s}_0}$. Moreover, by Theorem 7.5 of [**GHKK**], the vector space

$$\mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}) = \bigoplus_{q \in \mathcal{A}^{\vee}_{\mathrm{prin},\mathbf{s}_0}} \mathbb{C} \cdot \vartheta_q$$

has a $\mathbb{C}[N]$-algebra structure and we have canonical inclusion

$$\mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin},\mathbf{s}_0})$$

where $\mathrm{up}(\mathcal{A}_{\mathrm{prin},\mathbf{s}_0})$ is the ring of regular functions of $\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}$.

Proposition 2.4.4. *Given any* (m,n) *in* $\mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}_0}(\mathbb{Z}^T)$, *if for some generic point* Q *in* $\Delta^-_{\mathbf{s}_0}$, *there are finitely many broken lines with initial direction* (m,n) *and endpoints* Q, *then the same is true for any generic point* $Q^{'}$ *in* $\Delta^-_{\mathbf{s}_0}$. *In particular,* $\vartheta_{(m,n),Q'}$ *is a positive Laurent polynomial.*

Proof. This proposition follows from the arguments in Proposition 7.1 of [**GHKK**] with signs reversed. □

Definition 2.4.5. Let $\Theta^- \subset \mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}_0}(\mathbb{Z}^T)$ be the collection of (m,n) such that for some (or equivalently for any by Proposition 2.4.4) generic point Q in $\Delta^-_{\mathbf{s}_0}$, there are finitely many broken lines with initial direction (m,n) and end point Q.

Theorem 2.4.6. *If* $\mathcal{C}^+_{\mathbf{s}_0}$ *is contained in* Θ^-, *then we have the canonical inclusion*

$$\mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}) \subset \mathrm{up}(\tilde{\mathcal{A}}_{\mathrm{prin},\mathbf{s}_0})$$

where $\mathrm{up}(\tilde{\mathcal{A}}_{\mathrm{prin},\mathbf{s}_0})$ *is the ring of regular function of* $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbf{s}_0}$.

Proof. Since we already have the canonical inclusion $\mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbf{s}_0}) \subset \mathrm{up}(\mathcal{A}_{\mathrm{prin},\mathbf{s}_0})$, it suffices to show that for any (m,n) in $\mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}_0}$, for any generic point Q in $\Delta^-_{\mathbf{s}_0}$, $\vartheta_{(m,n),Q}$ is a Laurent polynomial. In particular, it suffices to show that $\Theta^- = \mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}_0}(\mathbb{Z}^T)$. First we show that $\mathcal{C}^-_{\mathbf{s}_0}$ is contained in Θ^-. Indeed, using the same arguments in Proposition 3.8 and Corollary 3.9 of [**GHKK**]with signs reversed, we can show that given a chamber σ in $\Delta^-_{\mathbf{s}_0}$ and a generic point in Q in $\mathrm{int}(\sigma)$, for initial direction (m,n) with (m,n) in $\sigma \bigcap (M^\circ \oplus N)$, there is only one broken line with end point Q. Next, we show that Θ^- is closed under addition. Indeed, if p,q are in Θ^-, consider the multiplication $\vartheta_p \cdot \vartheta_q$. Let $\alpha(p,q,p+q)$ be the coefficient of ϑ_{p+q} in the expansion by theta functions of $\vartheta_p \cdot \vartheta_q$. The pair of straight lines with initial directions p and q respectively and end point very close to $p+q$ will contribute to $\alpha(p,q,p+q)$, so we have $\alpha(p,q,p+q) \geqslant 1$. Write

$$\vartheta_p \cdot \vartheta_q = \sum_r \alpha(p,q,r)\vartheta_r$$

The right hand side is a finite sum with non-negative coefficients. Given a generic point Q in $\Delta^-_{\mathbf{s}_0}$, we have

$$\vartheta_{p,Q} \cdot \vartheta_{q,Q} = \sum_r \alpha(p,q,r)\vartheta_{r,Q}$$

Since each $\vartheta_{r,Q}$ is a sum of Laurent series with positive coefficients and each of $\vartheta_{q,Q}$ and $\vartheta_{p,Q}$ is a positive Laurent polynomial, we conclude that each $\vartheta_{r,Q}$ is also a positive Laurent polynomial. Hence for each r such that $\alpha(p,q,r) \neq 0$, r is contained in Θ^-. In particular, $p+q$ is contained in Θ^-. Since we have both $\mathcal{C}^+_{\mathbf{s}_0}$ and $\mathcal{C}^-_{\mathbf{s}_0}$ contained in Θ^- and Θ^- is closed under addition, we conclude that $\Theta^- = \mathcal{A}^\vee_{\mathrm{prin},\mathbf{s}_0}(\mathbb{Z}^T)$. $\square$

2.5. The Full Fock-Goncharov conjecture for $\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1}$, $\mathcal{X}_{\mathbb{T}^2_1}$ and $\mathcal{A}_{\mathbb{T}^2_1}$

2.5.1. The Full Fock-Goncharov conjecture for $\mathcal{A}_{\mathrm{prin}\mathbb{T}^2_1}$ and $\mathcal{X}_{\mathbb{T}^2_1}$.

In this section, we will study in detail cluster varieties associated to once punctured torus $\mathbb{T}^2_1$. To apply techniques we have developed so far, we will construct the seed associated to the ideal triangulation of $\mathbb{T}^2_1$ from that associated to the ideal triangulation of the four-punctured sphere $\mathbb{S}^2_4$. Consider the following ideal triangulation of $\mathbb{S}^2_4$:

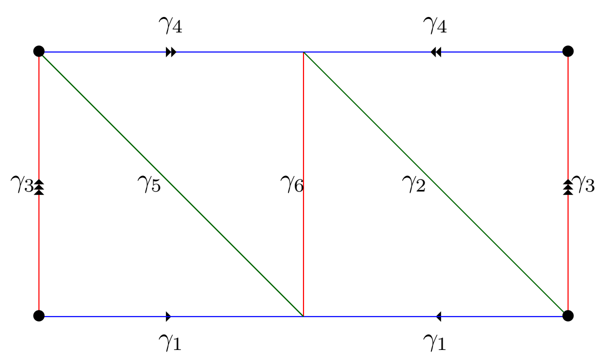


Figure 2.5.1. An ideal triangulation of the 4-punctured sphere $\mathbb{S}_4^2$

This ideal triangulation yields the following exchange matrix ϵ:

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Let N be a lattice of rank 6 with a basis $(e_i)_{i\in I}$, $I = \{1, \cdots, 6\}$. The exchange matrix ϵ equips N with the following skew-symmetric $\mathbb{Z}$-bilinear form:

$$\{,\} : N \times N \to \mathbb{Z}$$

$$\{e_i, e_j\} = \epsilon_{ij}$$

Let $\mathbf{s} = (e_i)_{i\in I}$ be our initial seed. Let $(f_i)_{i\in I}$ be the basis for the dual lattice M with $f_i = e_i^*$.

Let $\Pi =< \sigma_1 > \times < \sigma_2 > \times < \sigma_3 >$ be the subgroup of S_6 where σ_i is the involution that interchange the indices i and $i + 3$ for each $i = 1, 2, 3$. The group Π acts on the index set I for $\mathbf{s}$ via permutation of indices. Denote the orbit of i under the action of Π by $\bar{i}$. Applying the quotient construction in Section 2.3, we obtained the following data:

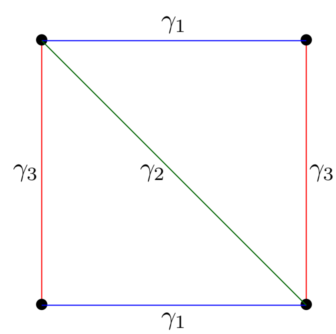
- An index set $\overline{I} = \{\bar{i}\}_{i=1,2,3}$.
- A lattice $\overline{N}$ of rank 3 with a basis $(e_{\bar{i}})_{\bar{i}\in\overline{I}}$ whose dual lattice $\overline{M}$ has a basis $(e^*_{\bar{i}})_{\bar{i}\in\overline{I}}$.
- A skew-symmetric bilinear form $\{,\} : \overline{N} \times \overline{N}\{,\} \to \mathbb{Z}$ on $\overline{N}$ such that $\{e_{\bar{i}}, e_{\bar{j}}\} = \{e_i, e_j\}$.
- For each $\bar{i} \in \overline{I}$, a positive integer $d_{\bar{i}} = 2$.

- A sublattice $\overline{N}^{\circ}$ of $\overline{N}$ with a basis $(d_{\bar{i}} e_{\bar{i}})_{\bar{i} \in \bar{I}}$ whose dual lattice $\overline{M}^{\circ}$ has a basis $(f_{\bar{i}})_{\bar{i} \in \bar{I}}$ where $f_{\bar{i}} = \frac{1}{d_{\bar{i}}} e^*_{\bar{i}}$.

The new seed $\bar{\mathbf{s}}$ is the labelled basis $(e_{\bar{i}})_{\bar{i} \in \bar{I}}$. The exchange matrix $\bar{\epsilon}$ for $\bar{\mathbf{s}}$ is as follows:

$$\begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

It is the exchange matrix arising from the following ideal triangulation of $\mathbb{T}^2_1$:



The quotient map $q : N_{\mathbb{R}} \twoheadrightarrow \overline{N}_{\mathbb{R}}$ is defined as follows:

$$\begin{aligned} q : N_{\mathbb{R}} &\twoheadrightarrow \overline{N}_{\mathbb{R}} \\ e_i &\mapsto e_{\bar{i}} \\ e_{i+3} &\mapsto e_{\bar{i}} \end{aligned}$$

The map q has the following natural section

$$\begin{aligned} s : \overline{N}_{\mathbb{R}} &\to N_{\mathbb{R}} \\ e_{\bar{i}} &\mapsto \frac{1}{d_{\bar{i}}}(e_i + e_{i+3}) \end{aligned}$$

Its restriction $s : \overline{N}^{\circ} \to N$, $d_{\overline{i}} e_{\overline{i}} \mapsto e_i + e_{i+3}$ induces a map s^* on the dual lattices

$$
\begin{aligned}
s^* : M &\to \overline{M}^{\circ} \\
f_i &\mapsto f_{\overline{i}} \\
f_{i+3} &\mapsto f_{\overline{i}} \quad i = 1, 2, 3
\end{aligned}
$$

Our goal is to show that full Fock-Goncharov conjecture holds for $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$, that is,

$$\Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}) = \mathcal{A}^{\vee}_{\mathrm{prin},\mathbb{T}_1^2}(\mathbb{Z}^T)$$

and

$$\mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}) = \Gamma(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}, \mathcal{O}_{\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}})$$

where $\mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2})$ is the $\mathbb{C}$-algebra with a vector space basis parametrized by $\Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2})$.

Our first step is to show that $\Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}) = \mathcal{A}^{\vee}_{\mathrm{prin},\mathbb{T}_1^2}(\mathbb{Z}^T)$. Let Q be a generic point in the interior of $\mathcal{C}^+_{\overline{\mathbf{s}}}$. It is known that the path-product $\mathfrak{p}^{\overline{\mathbf{s}}}_{-,+}$ is a composition of infinitely many wall-crossing automorphisms, so it is hard to do the computation directly. Instead, we apply the commutative diagram **??** to our case and obtain the following equality:

$$\mathfrak{p}^{\overline{\mathbf{s}}}_{-,+}\left((z^{(-f_{\overline{i}},0)})\right) = \phi_{-f_i} \circ \mathfrak{p}^{\mathbf{s}}_{-,+}(z^{(-f_i,0)})$$

Let $T_{N\oplus M}$ be the algebraic torus whose character lattice is $M \oplus N$. Recall the inversion map:

$$
\begin{aligned}
\Sigma : T_{N\oplus M} &\to T_{N\oplus M} \\
z^{(m,n)} &\mapsto z^{(-m,-n)}
\end{aligned}
$$

Then $\mathfrak{p}^{\mathbf{s}}_{-,+} \circ \Sigma^*$ is the *Donaldson-Thomas transformation* of $\mathcal{A}_{\mathrm{prin},\mathbb{S}^2_4}$ with respect to the seed $\mathbf{s}$, as defined in [**GS16**]. By the existence of maximal green sequence, DT transformation of $\mathcal{A}_{\mathrm{prin},\mathbb{S}^2_4}$ can be written as a composition of finitely many cluster mutations. In particular, $\mathfrak{p}^{\mathbf{s}}_{-,+} \circ \Sigma(z^{(f_i,0)}) = \mathfrak{p}^{\mathbf{s}}_{-,+}(z^{(-f_i,0)})$ is a positive Laurent polynomial. Hence after passing through the quotient map ϕ_{-f_i}, we conclude that $\mathfrak{p}^{\overline{\mathbf{s}}}_{-,+}\left((z^{(-f_{\bar{i}},0)})\right)$ is also a positive Laurent polynomial. Therefore, $(-f_{\bar{i}}, 0)$ is contained in $\Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1})$ for each $\bar{i}$. Since $\Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1})$ is saturated, closed under addition and translation by $\overline{N}$ after the identification $\mathcal{A}^\vee_{\mathrm{prin},\mathbb{T}^2_1}(\mathbb{Z}^T) \simeq \overline{M}^\circ \oplus \overline{N}$, we conclude that $\mathcal{C}^-_{\overline{\mathbf{s}}}(\mathbb{Z}^T)$ is contained in $\Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1})$. Since $\mathcal{C}^+_{\overline{\mathbf{s}}}(\mathbb{Z}^T)$ is also contained in $\Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1})$ and $\Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1})$ is closed under addition we conclude that $\mathcal{A}^\vee_{\mathrm{prin},\mathbb{T}^2_1}(\mathbb{Z}^T) = \Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1})$.

By Theorem 7.5 of [**GHKK**], $\mathrm{can}(\mathcal{A}_{\mathrm{prin}})$ is always contained in $\Gamma(\mathcal{A}_{\mathrm{prin}}, \mathcal{O}_{\mathcal{A}_{\mathrm{prin}}})$. To prove that $\mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1})$ is equal to $\Gamma(\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1}, \mathcal{O}_{\mathcal{A}_{\mathrm{prin},\mathbb{T}^2_1}})$, we need to do some more explicit computations. For our choice of initial seed $\mathbf{s}$, the following mutation sequence is a maximal green sequence:

$$[1, 3, 2, 4, 6, 5, 1, 6, 4, 3, 2, 5]$$

The DT-transformation $\mathfrak{p}^{\mathbf{s}}_{-,+} \circ \Sigma$ can be computed via compositions of cluster mutations along the above sequence of directions. For $i = 1, 2, 3$,

$$\begin{aligned}
\mathfrak{p}^{\overline{\mathbf{s}}}_{-,+}\left((z^{(-f_{\bar{i}},0)})\right) &= \phi_{-f_i} \circ \mathfrak{p}^{\mathbf{s}}_{-,+} \circ \Sigma(z^{(f_i,0)}) \\
&= \phi_{-f_i}\left[z^{(-f_i,0)} \cdot F\left(z^{(\{e_1,\cdot\},e_1)}, z^{(\{e_2,\cdot\},e_2)}, \cdots, z^{(\{e_6,\cdot\},e_6)}\right)\right] \\
&= z^{(-f_{\bar{i}},0)} \cdot \overline{F}\left(z^{(\{e_{\bar{i}},\cdot\},e_{\bar{i}})}, z^{(\{e_{\overline{i+1}},\cdot\},e_{\overline{i+1}})}, z^{(\{e_{\overline{i+2}},\cdot\},e_{\overline{i+2}})}\right)
\end{aligned}$$

Here,

$$F(x_1,x_2,\cdots,x_6)=1+x_1+x_1x_3+x_1x_6+x_1x_3x_6+x_1x_3x_5x_6$$
$$+x_1x_2x_3x_6+x_1x_2x_3x_5x_6+x_1^2x_2x_3x_5x_6$$
$$\overline{F}(y_1,y_2,y_3)=1+y_1+2y_1y_3+y_1y_3^2+2y_1y_2y_3^2+y_1y_2^2y_3^2+y_1^2y_2^2y_3^2$$

With an argument very similar to the case of the Kronecker 2 quiver, we can directly enumerate broken lines with initial direction $(f_{\bar{i}}-f_{\overline{i+1}},0)$ and ending at a generic point Q in $\mathcal{C}^+_{\bar{\mathbf{s}}}$. The theta function $\vartheta_{(f_{\bar{i}}-f_{\overline{i+1}},0)}$ has expansion at Q as follows:

$$\vartheta_{(f_{\bar{i}}-f_{\overline{i+1}},0),Q}=z^{(f_{\bar{i}}-f_{\overline{i+1}},0)}\left(1+z^{\left(\{e_{\overline{i+1}},\cdot\},e_{\overline{i+1}}\right)}+z^{\left(\{e_{\bar{i}}+e_{\overline{i+1}},\cdot\},e_{\bar{i}}+e_{\overline{i+1}}\right)}\right)$$

Indeed, given a broken line γ with initial direction $(f_{\bar{i}}-f_{\overline{i+1}},0)$, first observe that in order for γ to end at Q, the first bending of γ can only happen at the wall

$$\mathfrak{d}_1=\left(\mathbb{R}_{\geqslant 0}f_{\bar{i}}+\mathbb{R}_{\geqslant 0}f_{\overline{i+2}}+\overline{N}_{\mathbb{R}},1+z^{\left(\{e_{\overline{i+1}},\cdot\},e_{\overline{i+1}}\right)}\right)$$

or the wall

$$\mathfrak{d}_2=\left(\mathbb{R}_{\geqslant 0}\left(-f_{\bar{i}}+2f_{\overline{i+2}}\right)+\mathbb{R}_{\geqslant 0}f_{\overline{i+2}}+\overline{N}_{\mathbb{R}},1+z^{\left(\{e_{\overline{i+1}},\cdot\},e_{\overline{i+1}}\right)}\right)$$

Both $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are contained in the hyperplane $e^{\perp}_{\overline{i+1}}$. If γ has first bending anywhere else, it will shoot back out and never reach $\mathcal{C}^+_{\bar{\mathbf{s}}}$. Next, observe that by alternatively mutating in the direction $\bar{i}$ and $\overline{i+1}$, we obtain a copy $\mathfrak{D}$ of the scattering diagram of the Kronecker 2 quiver inside $\mathfrak{D}_{\bar{\mathbf{s}}}$ that contain $\mathfrak{d}_1$ and $\mathfrak{d}_2$. Thus, the enumeration of all broken lines with initial direction $\left(\left(f_{\bar{i}}-f_{\overline{i+1}}\right),0\right)$ and end point Q happens inside $\mathfrak{D}\subset\mathfrak{D}_{\bar{\mathbf{s}}}$ and can be reduced to the case of Example 3.10 in [**GHKK**].

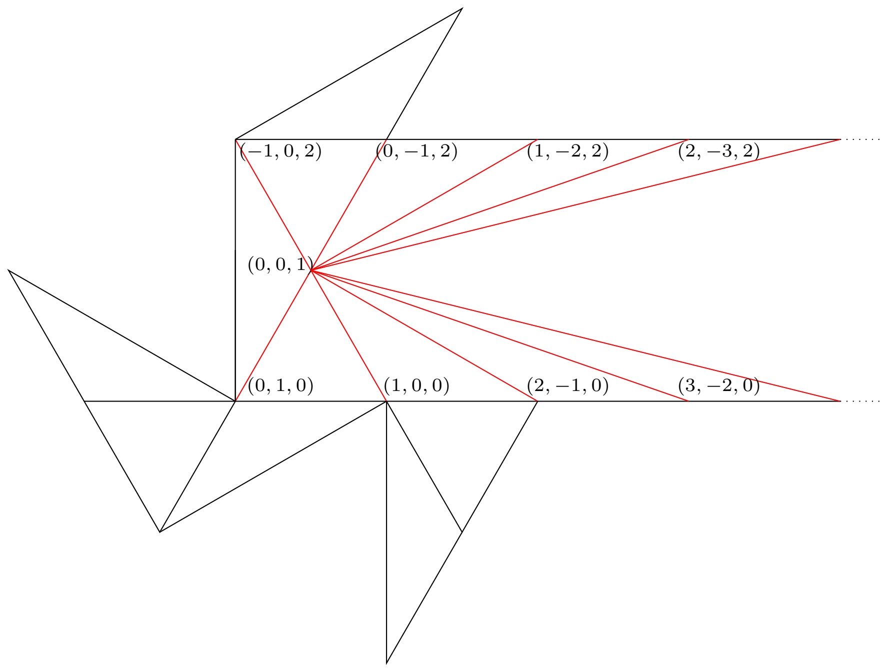


Figure 2.5.2. The projection of $\mathfrak{D}_{\overline{\mathbf{s}}}$ on to the plane $x+y+z=1$ in $\overline{M}_{\mathbb{R}}$. If we only mutate in the direction of $\overline{1}$ and $\overline{2}$, we obtain a copy of the scattering diagram of the Kronecker 2 quiver as shaded by red.

For any Q in $\mathfrak{D}_{\overline{\mathbf{s}}} \backslash \mathrm{supp}(\mathfrak{D}_{\overline{\mathbf{s}}})$ [4], for any $\overline{n}$ in $\overline{N}$, $\vartheta_{(0,\overline{n}),Q} = z^{(0,\overline{n})}$. Hence we will replace $\vartheta_{(0,\overline{n})}$ with $z^{(0,\overline{n})}$ for any $\overline{n}$ in $\overline{N}$. The upper cluster algebra $\Gamma(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}, \mathcal{O}_{\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}})$ is computed explicitly in [**MM13**] with a given set of generators. If we compare our computation with that of [**MM13**], after change of variables, we can easily see that the generators of $\Gamma(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}, \mathcal{O}_{\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}})$ given in [**MM13**] are all theta functions, that is, $\{\vartheta_{\pm(f_{\overline{i}},0)}, z^{\pm(0,e_{\overline{i}})}, \vartheta_{(f_{\overline{i}}-f_{\overline{i+1}},0)}\}_{i=1,2,3}$ is a set of generators for $\Gamma(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}, \mathcal{O}_{\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}})$. Thus we conclude that

$$\mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}) = \Gamma(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}, \mathcal{O}_{\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}})$$

Hence we prove the first main theorem of this section:

[4] Again, same as in section 4, we will simply denote $\mathfrak{D}_{\overline{\mathbf{s}}}^{\mathcal{A}_{\mathrm{prin}}}$ by $\mathfrak{D}_{\overline{\mathbf{s}}}$. For justification of this abuse of notation, see the discussion at the end of section 3

THEOREM 2.5.1. *Full Fock-Goncharov conjecture holds for* $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ *, that is,*

$$\Theta(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}) = \mathcal{A}^\vee_{\mathrm{prin},\mathbb{T}_1^2}(\mathbb{Z}^T), \quad \mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}) = \Gamma(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}, \mathcal{O}_{\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}})$$

The corollary below follows immediately:

COROLLARY 2.5.2. $\Theta(\mathcal{X}_{\mathbb{T}_1^2}) = \mathcal{X}^\vee_{\mathbb{T}_1^2}(\mathbb{Z}^T)$ *and* $\mathrm{can}(\mathcal{X}_{\mathbb{T}_1^2}) = \Gamma(\mathcal{X}_{\mathbb{T}_1^2}, \mathcal{O}_{\mathcal{X}_{\mathbb{T}_1^2}})$.

2.5.2. A cluster variety with two non-equivalent cluster structures. Given a lattice L, let $T_L = \mathbb{G}_m \otimes L$ be the algebraic torus whose cocharacter lattice is L. Given an initial seed $\mathbf{s}_0$, let $\mathfrak{T}_{\mathbf{s}_0}$ be the oriented tree whose root has $\mathbf{s}_0$ attached to it and whose vertices parametrize seeds mutationally equivalent to $\mathbf{s}_0$. There is an labelled edge $w \xrightarrow{k} w'$ in $\mathfrak{T}_{\mathbf{s}_0}$ if and only if $\mathbf{s}_w = \mu_k(\mathbf{s}_{w'})$. We write the cluster variety associated to $\mathbf{s}_0$ of type $\mathcal{A}$ or $\mathcal{X}$ or $\mathcal{A}_{\mathrm{prin}}$ as $V = \bigcup_{w\in\mathfrak{T}_{\mathbf{s}_0}} T_{L,\mathbf{s}_w}$ where the lattice L depends on the type of V.

DEFINITION 2.5.3. Given a S_2 scheme V', if there exists a cluster variety $V = \bigcup_{w\in\mathfrak{T}_{\mathbf{s}_0}} T_{L,\mathbf{s}_w}$ of type $\mathcal{A}$, $\mathcal{X}$ or $\mathcal{A}_{\mathrm{prin}}$ together with an open embedding

$$\iota : V \hookrightarrow V'$$

such that the complement of the image of V has codimension 2 in V', then we say that the torus charts $T_{L,\mathbf{s}_w}$ together with their embedding $\iota_{\mathbf{s}_w} : T_{L,\mathbf{s}_w} \hookrightarrow V'$ into V' provide *an atlas of cluster torus charts* for V'. Moreover, we say that the atlas $\{T_{L,\mathbf{s}_w}, \iota_{\mathbf{s}_w}\}_{w\in\mathfrak{T}_{\mathbf{s}_0}}$ has the same type as that of V.

DEFINITION 2.5.4. Given a S_2 scheme V, given two atlases of cluster torus charts

$$\mathcal{T}_1 = \{(T_{L,\mathbf{s}_w}, \iota_{\mathbf{s}_w})\}_{w\in\mathfrak{T}_{\mathbf{s}_0}}, \quad \mathcal{T}_2 = \left\{\left(T_{L',\mathbf{s}_v}, \iota_{\mathbf{s}_v}\right)\right\}_{v\in\mathfrak{T}_{\mathbf{s}'_0}}$$

of the same type for V, we say $\mathcal{T}_1$ and $\mathcal{T}_2$ give two *non-equivalent cluster structures* for V if for any w in $\mathfrak{T}_{\mathbf{s}_0}$, there is no v in $\mathfrak{T}_{\mathbf{s}_0'}$ such that the embeddings $\iota_{\mathbf{s}_w} : T_{L,\mathbf{s}_w} \hookrightarrow V$ and $\iota_{\mathbf{s}_v} : T_{L,\mathbf{s}_v} \hookrightarrow V$ have the same image in V.

Definition 2.5.5. Given a variety V and an atlas of cluster torus charts $\mathcal{T} = \{(T_{L,\mathbf{s}_w}, \iota_{\mathbf{s}_w})\}_{w\in\mathfrak{T}_{\mathbf{s}_0}}$ for V, a *global monomial* for $\mathcal{T}$ is a regular function on V that restricts to a character on some torus $T_{L,\mathbf{s}_w}$ in $\mathcal{T}$.

Given our initial seed $\bar{\mathbf{s}}$ for $\mathcal{A}_{\mathbb{T}_1^2}$ as constructed in the beginning of this section, let $\mathfrak{T}_{\bar{\mathbf{s}}}$ be the infinite directed tree whose vertices parametrize seeds mutationally equivalent to $\bar{\mathbf{s}}$. Since $\mathfrak{p}^{\bar{\mathbf{s}}}_{+,-}$ is rational as we have shown, we can run the construction in Section 2.4 and build a variety $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2} := \tilde{\mathcal{A}}_{\mathrm{prin},\bar{\mathbf{s}}}$ that is isomorphic to the space $\tilde{\mathcal{A}}^{\mathrm{scat}}_{\mathrm{prin},\bar{\mathbf{s}}}$ we build using all chambers in $\Delta^+_{\bar{\mathbf{s}}}$ and $\Delta^-_{\bar{\mathbf{s}}}$ in the scattering diagram. Let $\mathcal{A}^{\pm}_{\mathrm{prin},\mathbb{T}_1^2} \subset \tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ be the copy of $\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}$ corresponding to $\Delta^{\pm}_{\bar{\mathbf{s}}}$ respectively.

Proposition 2.5.6. *We have*

$$\mathrm{up}(\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}) = \mathrm{can}(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2})$$

where $\mathrm{up}(\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2})$ *is the ring of regular functions of* $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$.

Proof. Since

$$\mathrm{up}\left(\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}\right) \subset \mathrm{up}\left(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}\right) = \mathrm{can}\left(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}\right)$$

to prove the equality, we only need to show that

$$\mathrm{can}\left(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2}\right) \subset \mathrm{up}\left(\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}\right)$$

By Theorem 2.4.6, it suffices to show that $\mathcal{C}^+_{\bar{\mathbf{s}}}$ is contained in $\Theta^-(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2})$. Indeed, given a generic point Q in $\mathcal{C}^-_{\bar{\mathbf{s}}}$, by direct computations we obtain that

$$\vartheta_{(f_{\overline{i}},0),Q} = \mathfrak{p}^{\overline{\mathbf{s}}}_{+,-}\left(z^{(f_{\overline{i}},0)}\right)$$
$$= z^{(f_{\overline{i}},0)}\overline{G}(z^{(\{e_{\overline{i}},\cdot\},e_{\overline{i}})}, z^{(\{e_{\overline{i+1}},\cdot\},e_{\overline{i+1}})}, z^{(\{e_{\overline{i+2}},\cdot\},e_{\overline{i+2}})})$$

where

$$G(y_1,y_2,y_3) = 1 + y_1 + 2y_1y_2 + y_1y_2^2 + 2y_1y_2^2y_3 + y_1y_2^2y_3^2 + y_1^2y_2^2y_3^2$$

Since $\Theta^-(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2})$ is closed under addition, as we have shown in the proof of 2.4.6, and translation by $\overline{N}$, we conclude that $\mathcal{C}^+_{\overline{\mathbf{s}}}$ is contained in $\Theta^-(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2})$. $\square$

It follows immediately from Proposition 2.5.6 that $\mathrm{up}(\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}) = \mathrm{up}(\mathcal{A}_{\mathrm{prin},\mathbb{T}_1^2})$. Hence, each of $\mathcal{A}^{\pm}_{\mathrm{prin},\mathbb{T}_1^2}$ has codimension 2 inside $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ and cluster tori for each of $\mathcal{A}^{\pm}_{\mathrm{prin},\mathbb{T}_1^2}$ provide an atlas of cluster torus charts for $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$. We denote these two atlases by $\mathcal{T}^+$ and $\mathcal{T}^-$ respectively.

THEOREM 2.5.7. *The atlases of cluster torus charts $\mathcal{T}^+$ and $\mathcal{T}^-$ give $\tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$ two non-equivalent cluster structures in the sense of 2.5.3, corresponding to $\Delta^+_{\overline{\mathbf{s}}}$ and $\Delta^-_{\overline{\mathbf{s}}}$ respectively.*

PROOF. Write

$$\mathcal{T}^{\pm} = \left\{T^{\pm}_{\overline{N}^\circ\oplus\overline{M},\overline{\mathbf{s}}_v}, \iota^{\pm}_v\right\}_{v\in\mathfrak{T}_{\overline{\mathbf{s}}}}$$

Given any v in $\mathfrak{T}_{\overline{\mathbf{s}}}$, assume there exists w in $\mathfrak{T}_{\overline{\mathbf{s}}}$ such that the embeddings

$$\iota^+_v : T^+_{\overline{N}^\circ\oplus\overline{M},\overline{\mathbf{s}}_v} \hookrightarrow \tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$$

and

$$\iota^-_w : T^-_{\overline{N}^\circ\oplus\overline{M},\overline{\mathbf{s}}_w} \hookrightarrow \tilde{\mathcal{A}}_{\mathrm{prin},\mathbb{T}_1^2}$$

have the same image. Characters on $T^+_{\overline{N}^\circ\oplus\overline{M},\overline{\mathbf{s}}_v}$ that are global monomials for $\mathcal{T}^+$ (cf. Definition 2.5.5) correspond to theta functions parametrized by

$\mathcal{C}_v^+(\mathbb{Z}^T)$. Characters of $T^-_{\overline{N}^\circ \oplus \overline{M}, \overline{\mathbf{s}}_w}$ that are regular functions on $\tilde{\mathcal{A}}_{\mathrm{prin}, \mathbb{T}_1^2}$ correspond to theta functions parametrized by $\mathcal{C}_w^-(\mathbb{Z}^T)$. Since $\mathcal{C}_w^-(\mathbb{Z}^T) \bigcap \mathcal{C}_v^+(\mathbb{Z}^T) = 0 \oplus \overline{N}$, no characters other than $z^{(0,\overline{n})}$ for $\overline{n} \in \overline{N}$ on $T^-_{\overline{N}^\circ \oplus \overline{M}, \overline{\mathbf{s}}_w}$ can be global monomials for $\mathcal{T}^+$, which is in contradiction with that the embeddings ι_v^+ and ι_w^- have the same image. Hence $\mathcal{T}^+$ and $\mathcal{T}^-$ give $\tilde{\mathcal{A}}_{\mathrm{prin}, \mathbb{T}_1^2}$ two non-equivalent cluster structures. $\square$

2.5.3. The Full Fock-Goncharov conjecture for $\mathcal{A}_{\mathbb{T}_1^2}$. Since $\Theta(\mathcal{A}_{\mathrm{prin}, \mathbb{T}_1^2}) = \mathcal{A}^\vee_{\mathrm{prin}, \mathbb{T}_1^2}$, it follows that $\Theta(\mathcal{A}_{\mathbb{T}_1^2}) = \mathcal{A}^\vee_{\mathbb{T}_1^2}$. However, we will see that $\mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2})$ is strictly contained in $\Gamma(\mathcal{A}_{\mathbb{T}_1^2}, \mathcal{O}_{\mathbb{T}_1^2})$. Hence the Full Fock-Goncharov conjecture does not hold for $\mathcal{A}_{\mathbb{T}_1^2}$. Since

$$\{\vartheta_{(\pm f_{\overline{i}}, 0)}, \vartheta_{(f_{\overline{i}-\overline{i+1}}, 0)}, z^{(0, \pm e_{\overline{i}})}\}_{i=1,2,3}$$

is a set of generators of $\mathrm{can}(\mathcal{A}_{\mathrm{prin}, \mathbb{T}_1^2})$, $\{\vartheta_{\pm f_{\overline{i}}}, \vartheta_{f_{\overline{i}-\overline{i+1}}}\}_{i=1,2,3}$ is a set of generators of $\mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2})$. Denote $z^{f_{\overline{i}}}$ by A_i. Then with respect to the initial seed $\overline{\mathbf{s}}$, $\mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2})$ has the following set of generators:

$$A_i, \quad \frac{A_1^2 + A_2^2 + A_3^2}{A_{i+1}A_{i+2}} = A_i \left(\frac{A_1^2 + A_2^2 + A_3^2}{A_1 A_2 A_3} \right),$$

$$\frac{(A_1^2 + A_2^2 + A_3^2)^2}{A_i A_{i+1}^2 A_{i+2}^2} = A_i \left(\frac{A_1^2 + A_2^2 + A_3^2}{A_1 A_2 A_3} \right)^2, \quad i = 1, 2, 3 \mod 3$$

By [**MM13**], $\Gamma(\mathcal{A}_{\mathbb{T}_1^2}, \mathcal{O}_{\mathbb{T}_1^2})$ has the following set of generators:

$$\left\{ A_1, A_2, A_3, \frac{A_1^2 + A_2^2 + A_3^2}{A_1 A_2 A_3} \right\}$$

Hence $\mathrm{can}\left(\mathcal{A}_{\mathbb{T}_1^2}\right)$ is strictly contained in $\Gamma(\mathcal{A}_{\mathbb{T}_1^2}, \mathcal{O}_{\mathbb{T}_1^2})$.

We can run the construction in Section 2.4 similarly for $\mathcal{A}_{\mathbb{T}_1^2}$ and obtain a new variety $\tilde{\mathcal{A}}_{\mathbb{T}_1^2} \supset \mathcal{A}_{\mathbb{T}_1^2}$. Moreover, given each oriented edge $v \xrightarrow{k} v'$ in $\mathfrak{T}_{\overline{\mathbf{s}}}$, the commutative diagram

$$\begin{tikzcd}
T^{\pm}_{\overline{N}^\circ \oplus \overline{M}, \bar{\mathbf{s}}_v} \arrow[rr, dashed, "\mu_k"] \arrow[dr, hook] & & T^{\pm}_{\overline{N}^\circ \oplus \overline{M}, \bar{\mathbf{s}}_{v'}} \arrow[dl, hook'] \\
& \widetilde{\mathcal{A}}_{\mathrm{prin}, \mathbb{T}_1^2} \arrow[d, "\tilde{\pi}"] & \\
& T_{\overline{M}} &
\end{tikzcd}$$

restrict to

$$\begin{tikzcd}
T^{\pm}_{\overline{N}^\circ, \bar{\mathbf{s}}_v} \arrow[rr, dashed, "\mu_k"] \arrow[dr, hook] & & T^{\pm}_{\overline{N}^\circ, \bar{\mathbf{s}}_{v'}} \arrow[dl, hook'] \\
& \widetilde{\mathcal{A}}_{\mathbb{T}_1^2} \arrow[d, "\tilde{\pi}"] & \\
& e &
\end{tikzcd}$$

Given each vertex v, we can compute the gluing map $T^{+}_{\overline{N}^\circ, \bar{\mathbf{s}}_v} \dashrightarrow T^{-}_{\overline{N}^\circ, \bar{\mathbf{s}}_v}$ very explicitly. For the initial seed $\bar{\mathbf{s}}$, the gluing map $\alpha_{\bar{\mathbf{s}}} : T^{+}_{\overline{N}^\circ, \bar{\mathbf{s}}} \dashrightarrow T^{-}_{\overline{N}^\circ, \bar{\mathbf{s}}}$ is given by

$$\alpha^*_{\bar{\mathbf{s}}}(A_i) = A_i \left(\frac{A_1^2 + A_2^2 + A_3^2}{A_1 A_2 A_3} \right)^2$$

For any seed $\bar{\mathbf{s}}_v = (e^{'}_{\bar{i}})_{\bar{i} \in \bar{I}}$ for v in $\mathfrak{T}_{\bar{\mathbf{s}}}$, denote $\frac{e^{'*}_{\bar{i}}}{2}$ by $A_i^{\bar{\mathbf{s}}_v}$. Let η be the rational function such that $\eta(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3}$. We know that $\eta(A_1, A_2, A_3)$ is invariant under cluster mutations, that is, for any seed $\bar{\mathbf{s}}^{'}$, we have

$$\eta(A_1^{\bar{\mathbf{s}}_v}, A_2^{\bar{\mathbf{s}}_v}, A_3^{\bar{\mathbf{s}}_v}) = \eta(A_1, A_2, A_3) = \eta$$

For each v in $\mathfrak{T}_{\bar{\mathbf{s}}}$, define the following birational map $\alpha_{\bar{\mathbf{s}}_v} : T_{\overline{N}^\circ, \bar{\mathbf{s}}} \dashrightarrow T_{\overline{N}^\circ, \bar{\mathbf{s}}}$ such that $\alpha^*_{\bar{\mathbf{s}}_v}(A_i^{\bar{\mathbf{s}}_v}) = A_i^{\bar{\mathbf{s}}_v} \cdot \eta^2$. We show that the birational maps $\alpha_{\bar{\mathbf{s}}_v}$ give the glue maps $T^{+}_{\overline{N}^\circ, \bar{\mathbf{s}}_v} \dashrightarrow T^{-}_{\overline{N}^\circ, \bar{\mathbf{s}}_v}$. It suffices to show that the following diagram commutes for any oriented edge $v \xrightarrow{k} v'$ in $\mathfrak{T}_{\bar{\mathbf{s}}}$:

$$\begin{tikzcd}
T_{\overline{N}^\circ, \bar{\mathbf{s}}_v} \arrow[r, dashed, "\mu_k"] \arrow[d, dashed, "\alpha_{\bar{\mathbf{s}}_v}"] & T_{\overline{N}^\circ, \bar{\mathbf{s}}_{v'}} \arrow[d, dashed, "\alpha_{\bar{\mathbf{s}}_{v'}}"] \\
T_{\overline{N}^\circ, \bar{\mathbf{s}}_v} \arrow[r, dashed, "\mu_k"] & T_{\overline{N}^\circ, \bar{\mathbf{s}}_{v'}}
\end{tikzcd}$$

Indeed, for $i = k$,

$$\begin{aligned}
\alpha^*_{\bar{\mathbf{s}}_v} \circ \mu_k^*(A_i^{\bar{\mathbf{s}}_{v'}}) &= \alpha^*_{\bar{\mathbf{s}}_v}\left[\frac{\left(A_{k+1}^{\bar{\mathbf{s}}_{v'}}\right)^2 + \left(A_{k+2}^{\bar{\mathbf{s}}_{v'}}\right)^2}{A_k^{\bar{\mathbf{s}}_{v'}}}\right] \\
&= \frac{\left(A_{k+1}^{\bar{\mathbf{s}}_v} \cdot \eta^2\right)^2 + \left(A_{k+2}^{\bar{\mathbf{s}}_v} \cdot \eta^2\right)^2}{A_k^{\bar{\mathbf{s}}_v} \cdot \eta^2} \\
&= \frac{\left(A_{k+1}^{\bar{\mathbf{s}}_v}\right)^2 + \left(A_{k+2}^{\bar{\mathbf{s}}_v}\right)^2}{A_k^{\bar{\mathbf{s}}_v}} \cdot \eta^2 \\
&= \mu_k^*\left(A_k^{\bar{\mathbf{s}}_{v'}}\right)\mu_k^*\left(\eta^2\right) \\
&= \mu_k^*\left(A_k^{\bar{\mathbf{s}}_{v'}} \cdot \eta^2\right) \\
&= \mu_k^* \circ \alpha^*_{\bar{\mathbf{s}}_{v'}}\left(A_k^{\bar{\mathbf{s}}_{v'}}\right)
\end{aligned}$$

For $i \neq k$,

$$\alpha^*_{\bar{\mathbf{s}}_v} \circ \mu_k^*(A_i^{\bar{\mathbf{s}}_{v'}}) = \alpha^*_{\bar{\mathbf{s}}_v}(A_i^{\bar{\mathbf{s}}_v}) = A_i^{\bar{\mathbf{s}}_v} \cdot \eta^2 = \mu_k^*\left(A_i^{\bar{\mathbf{s}}_{v'}}\right) \cdot \mu_k^*(\eta^2) = \mu_k^* \circ \alpha^*_{\bar{\mathbf{s}}_{v'}}\left(A_i^{\bar{\mathbf{s}}_{v'}}\right)$$

Hence the diagram commutes.

Having already written down explicitly the gluing maps for $\tilde{\mathcal{A}}_{\mathbb{T}_1^2}$, let us show that

$$\Gamma(\tilde{\mathcal{A}}_{\mathbb{T}_1^2}, \mathcal{O}_{\tilde{\mathcal{A}}_{\mathbb{T}_1^2}}) = \mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2})$$

Indeed, consider $\alpha_{\bar{\mathbf{s}}}^{-1} : T^-_{\overline{N},\bar{\mathbf{s}}} \dashrightarrow T^+_{\overline{N},\bar{\mathbf{s}}}$. Since $\left(\alpha_{\bar{\mathbf{s}}}^{-1}\right)^*(\eta) = \eta^{-1}$, $\eta = \frac{A_1^2+A_2^2+A_3^2}{A_1A_2A_3}$ is not contained in $\Gamma(\tilde{\mathcal{A}}_{\mathbb{T}_1^2}, \mathcal{O}_{\tilde{\mathcal{A}}_{\mathbb{T}_1^2}})$. Recall that $\mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2})$ has the following set of generators:

$$\{A_i, A_i^{-1}\eta, A_i^{-1}\eta^2\}_{i=1,2,3}$$

Therefore $\Gamma(\tilde{\mathcal{A}}_{\mathbb{T}_1^2}, \mathcal{O}_{\tilde{\mathcal{A}}_{\mathbb{T}_1^2}})$ is contained in $\mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2})$. Since we also have the reverse containment $\mathrm{can}(\mathcal{A}_{\mathbb{T}_1^2}) \subset \Gamma(\tilde{\mathcal{A}}_{\mathbb{T}_1^2}, \mathcal{O}_{\tilde{\mathcal{A}}_{\mathbb{T}_1^2}})$, we conclude that they are equal. Since $\tilde{\mathcal{A}}_{\mathbb{T}_1^2}$ is the space where Fock-Goncharov duality conjecture holds not $\mathcal{A}_{\mathbb{T}_1^2}$, it justifies our motto that to build up the cluster variety, we should also include torus charts coming from negative chambers in the scattering diagram into the atlas of cluster torus charts.

CHAPTER 3

Admissible groups and cluster structures of families of Log Calabi-Yau surfaces

3.1. Introduction

In Chapter 3 of the thesis, we apply the folding technique as developed in Chapter 2 to families of log Calabi-Yau surfaces and their degenerate subfamilies, generalizing the key example we investigate in Chapter 2. First, we describe the action of the admissible groups on the scattering diagrams for cluster varieties associated to universal families of log Calabi-Yau surfaces in positive cases. The folding technique we developed in Chapter 2 gives us a functorial construction of degenerate subfamilies in universal families of log Calabi-Yau surfaces. In positive cases, the folding procedure also allows us to study the full theta basis and scattering diagrams of degenerate subfamilies, overcoming the technical difficulty of infinite wall-crossings that appear when we restrict to degenerate loci. As a byproduct, we construct examples of cluster varieties with non-equivalent cluster structures.

3.2. Classification of Looijenga pairs.

3.2.1. The positive cases. The following proposition follows from Section 4.3 of [**TM**]:

Proposition 3.2.1. *Let (Y, D) be a positive Looijenga pair and $\mathcal{X}$ the cluster variety associated to $U = Y \backslash D$. If $\mathcal{X}$ cannot be constructed from an* *acyclic seed, then*

$$D^{\perp} \simeq \mathbf{D}_n (n \geqslant 4) \text{ or } \mathbf{E}_n (n = 6, 7, 8)$$

Moreover, (Y, D) *can obtained from the toric model* $(\mathbb{P}^2, \overline{D})$ *where* $\overline{D} = \overline{D}_1 + \overline{D}_2 + \overline{D}_3$ *is the toric boundary divisor.*

3.2.2. The strictly negative semidefinite cases.

PROPOSITION 3.2.2. *If* D *is minimal, then the following are equivalent:*

(1) (Y, D) *is strictly negative semidefinite.*

(2) Either D *is irreducible with* $D^2 = 0$ *or* $D_i^2 = -2$ *for all* i*.*

(3) (Y, D) *is deformation equivalent to a Looijenga pair* $(Y^{'}, D^{'})$ *that admits an elliptic fibration* $Y^{'} \to \mathbb{P}^2$ *with* $D^{'}$ *a fiber.*

PROOF. We first prove that (1) $\Longleftrightarrow$ (2). Suppose (Y, D) is strictly negative semidefinite and D is reducible. Assume that $D_i^2 > -2$ for some i. Then by minimality of D, $D_i \geqslant 0$ and

$$\left[(-D_{i+1}^2 + 1)D_i + D_{i+1}\right]^2 \geqslant 2 - D_{i+1}^2 > 0$$

which is in contradiction with that (Y, D) is strictly negative semidefinite. Therefore the assumption is false and $D_i^2 \leqslant -2$ for all i. Since (Y, D) is not negative definite, by section 4.1 of [**TM**], we must also have $D_i^2 > -3$ for all i. Therefore $D_i^2 = -2$ for all i. Hence (1) implies (2). Now suppose that D is reducible and $D_i^2 = -2$ for all i. Then

$$\left(\sum_{i=1}^{n} a_i D_i\right)^2 = -\sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2$$

Therefore (2) implies (1).

Next, we prove that (2) $\Longleftrightarrow$ (3). If (3) holds, then $D \cdot D_i = 0$ for all i, which implies (2). Conversely, if (2) holds, let σ be an exceptional curve that intersects a boundary component D_i of D. Blow down σ and let $\overline{D}$ and $\overline{D}_i$ be the pushforward of D and D_i for all i. Then $\overline{D}^2 = \overline{D} \cdot D_i = 1$, $\overline{D} \cdot \overline{D}_j = 0$ for all $j \neq i$. By Riemann-Roch,

$$h^0(\overline{D}) - h^1(\overline{D}) + h^0(-2\overline{D}) = \chi(\overline{D}) = 1 + \overline{D}^2 = 2$$

Since $h^0(-2\overline{D}) = 0$, $h^0(\overline{D}) \geqslant 2$. Hence $\dim |\overline{D}| \geqslant 1$. Let $C \in |\overline{D}|$ be a curve linearly equivalent to $\overline{D}$ to not equal $\overline{D}$. Then $C \cdot \overline{D}_i = 1$ and $C \cdot \overline{D}_j = 0$ for $j \neq i$. Let p be the intersection of $\overline{D}_i$ and C. If we blow up at p, the new Looijenga pair $(Y^{'}, D^{'})$ we get will be deformation equivalent to (Y, D). Moreover, $|D^{'}|$ contains a pencil (the blowup of the pencil containing C and $\overline{D}$) that yields a fibration to $\mathbb{P}^1$ with $D^{'}$ a fiber. By the adjunction formula, the generic smooth fiber of the fibration $(Y^{'}, D^{'}) \to \mathbb{P}^1$ will have genus 1. Thus, this fibration is an elliptic fibration. $\square$

Let (Y, D) be a strictly negative semidefinite Looijenga pair with D minimal and reducible. By Proposition 3.2.2, $D_i^2 = -2$ for all i. Let σ_1 be an exceptional curve that intersects a boundary component of D. Up to reordering the labelling, we could assume that σ_1 intersects D_1. Then

$$H^2(Y, \mathbb{Z}) \simeq \mathbb{Z}[D] \oplus \mathbb{Z}[\sigma] \oplus \{[D], [\sigma]\}^{\perp}$$

where

$$\{[D], [\sigma]\}^{\perp} \simeq -E_8$$

Proposition 3.2.3. *If (Y, D) is strictly negative semidefinite with D minimal and reducible, then*

$$D^{\perp} = \mathbb{Z}[D] \oplus \left[\{[D], [\sigma_1]\}^{\perp} / \langle [D_i] \rangle_{i=2}^{n}\right]$$

as orthogonal direct sum. Here, $\langle [D_i] \rangle_{i=2}^{n}$ is the subgroup in $\{[D], [\sigma_1]\}^{\perp}$ generated by images of $[D_i]$ with $2 \leqslant i \leqslant n$.

3.3. The geometry of folding

First, we want to see how our quotient construction behave with respect to cluster mutations. Given a k in I, let $\{l_1, \cdots, l_{|\Pi k|}\}$ be an enumeration of elements of Πk.

LEMMA 3.3.1. *Let* $\mathbf{s}' = (e_i')_{i \in I} = \mu_{l_1} \circ \cdots \circ \mu_{l_k}(\mathbf{s})$. *Then* $\mathbf{s}'$ *does not depend on the choice of ordering of elements in* Πk. *With respect to the new seed* $\mathbf{s}'$, *the action of* Π *still satisfies the condition for folding, that is, for any* π_1, π_2 *in* Π,

$$\{e'_{\pi_1 \cdot i}, e'_{\pi_2 \cdot j}\} = \{e'_i, e'_j\}$$

Moreover, $\mu_{\Pi k}(\overline{\mathbf{s}}) = (q(e_i'))_{\Pi i \in \overline{I}}$ *where* $q : N \twoheadrightarrow \overline{N}$ *is the quotient map of lattice sending* e_i *to* $e_{\Pi i}$.

PROOF. First observe that for any $1 \leqslant a \leqslant k$, $e'_{l_a} = -e_{l_a}$. Therefore

$$\{e'_{l_{a_1}}, e'_{l_{a_2}}\} = 0$$

1) If $i, j \in I$ are not in Πk, for any π_1, π_2 in Π,

$$\begin{aligned} e'_{\pi_1 \cdot i} &= e_{\pi_1 \cdot i} + \sum_{a=1}^{|\Pi k|} [\{e_{\pi_1 \cdot i}, e_{l_a}\}]_+ e_{l_a} \\ &= e_{\pi_1 \cdot i} + [\{e_i, e_k\}]_+ \sum_{a=1}^{|\Pi k|} e_{l_a} \end{aligned}$$

Similarly, we have

$$e'_{\pi_2 \cdot j} = e_{\pi_2 \cdot j} + [\{e_j, e_k\}]_+ \sum_{a=1}^{|\Pi k|} e_{l_a}$$

Therefore,

$$\begin{aligned}
&\{e^{'}_{\pi_1\cdot i}, e^{'}_{\pi_2\cdot j}\}\\
=&\{e_{\pi_1\cdot i} + [\{e_i, e_k\}]_+ \sum_{a=1}^{|\Pi k|} e_{l_a},\ e_{\pi_2\cdot j} + [\{e_j, e_k\}]_+ \sum_{a=1}^{|\Pi k|} e_{l_a}\}\\
=&\{e_{\pi_1\cdot i}, e_{\pi_2\cdot j}\} + [\{e_i, e_k\}]_+\{\sum_{a=1}^{|\Pi k|} e_{l_a}, e_{\pi_2\cdot j}\} + [\{e_j, e_k\}]_+\{e_{\pi_1\cdot i}, \sum_{a=1}^{|\Pi k|} e_{l_a}\}\\
=&\{e_i, e_j\} + |\Pi k|\cdot[\{e_i, e_k\}]_+\cdot\{e_k, e_j\} + |\Pi k|\cdot[\{e_j, e_k\}]_+\{e_i, e_k\}\\
=&\{e^{'}_i, e^{'}_j\}
\end{aligned}$$

2) If $i \in I$ is not in Πk, for any $1 \leqslant a \leqslant |\Pi k|$,

$$\begin{aligned}
\{e^{'}_{\pi_1\cdot i}, e^{'}_{l_a}\} &= \{e_{\pi_1\cdot i} + [\{e_i, e_k\}]_+ \sum_{a=1}^{|\Pi k|} e_{l_a}, -e_{l_a}\}\\
&= \{e_i, -e_k\}\\
&= \{e^{'}_i, e^{'}_k\}
\end{aligned}$$

Hence the action of Π on I with respect to the new seed $\mathbf{s}^{'}$ still satisfies the folding condition. It remains to prove the second part of the lemma. Notice that for any $1 \leqslant a \leqslant k$, $q(e^{'}_{l_a}) = -e_{\Pi k} = \mu_{\Pi k}(e_{\Pi k})$. If $i \in I$ is not in Πk, for any π in Π,

$$\begin{aligned}
q(e^{'}_{\pi\cdot i}) &= q\left[e_{\pi\cdot i} + [\{e_i, e_k\}]_+ \sum_{a=1}^{|\Pi k|} e_{l_a}\right]\\
&= e_{\Pi i} + |\Pi k|[\{e_i, e_k\}]_+ e_{\Pi k}\\
&= e_{\Pi i} + |\Pi k|[\{e_{\Pi i}, e_{\Pi k}\}]_+ e_{\Pi k}\\
&= \mu_{\Pi k}(e^{'}_{\Pi i})
\end{aligned}$$

$\square$

Let w_0, v_0 be the root of $\mathfrak{T}_{\overline{\mathbf{s}}}$ and $\mathfrak{T}_{\mathbf{s}}$ respectively. Let $V(\mathfrak{T}_{\mathbf{s}})$ and $V(\mathfrak{T}_{\overline{\mathbf{s}}})$ be the set of vertices of $\mathfrak{T}_{\mathbf{s}}$ and $\mathfrak{T}_{\overline{\mathbf{s}}}$ respectively. For each orbit Πk, fix an enumeration $\{l_1, \cdots, l_{|\Pi k|}\}$of elements in Πk. We can construct an injective function

$$\delta : V(\mathfrak{T}_{\overline{\mathbf{s}}}) \hookrightarrow V(\mathfrak{T}_{\mathbf{s}})$$

such that $\rho(w_0) = v_0$ and for each oriented edge $w \overset{\Pi k}{\to} w'$ in $\mathfrak{T}_{\overline{\mathbf{s}}}$, we have the following path in $\mathfrak{T}_{\mathbf{s}}$:

$$\delta(w) \overset{l_1}{\to} v_1 \overset{l_2}{\to} \cdots \overset{l_{|\Pi k|}}{\to} \delta(w')$$

By Theorem 3.3.1, given any vertex w in $V(\mathfrak{T}_{\overline{\mathbf{s}}})$, the seed $\mathbf{s}_{\delta(w)} \in [\mathbf{s}]$ is independent of the choice of the enumeration of orbits of I. By abusing our notation, we omit ρ and denote $\mathbf{s}_{\delta(w)}$ by $\mathbf{s}_w$. Denote by $[\mathbf{s}]^{\mathrm{iso}}$ the subset of $[\mathbf{s}]$ consisting of seeds isomorphic to $\mathbf{s}$. Given $\mathbf{s}_v \in [\mathbf{s}]^{\mathrm{iso}}$, let $\Delta^+_{\mathbf{s}_v,\Pi}$ be the collection of chambers obtained via Π-constrained mutation, that is, if we mutate in the direction i, we must also mutate in other directions in the orbit Πi. Define

$$\mathcal{A}_{\mathrm{prin},\Delta^+_{\mathbf{s}_v,\Pi}} = \bigcup_{\sigma \in \Delta^+_{\mathbf{s}_v,\Pi}} T_{N\oplus M,\sigma}$$

where the gluing map between tori attached to different chambers in $\Delta^+_{\mathbf{s}_v,\Pi}$ are given by the birational wall-crossing automorphisms between them in $\mathfrak{D}_{\mathbf{s}}$.

Definition 3.3.2. Given a log Calabi-Yau variety U, we say that U has a *cluster structure in the weak sense* if there exists an atlas of cluster torus charts $\mathcal{T} = \{T_{L,\mathbf{s}} \mid \mathbf{s} \in \mathfrak{T}_{\mathbf{s}_0}\}$ together with an open immersion

$$\bigcup_{\mathbf{s} \in \mathfrak{T}_{\mathbf{s}_0}} T_{L,\mathbf{s}} \hookrightarrow U$$

such that the complement is of codimension 1 inside U.

Lemma 3.3.3. *Let* $\mathbf{s}_v$ *be an element in* $[\mathbf{s}]^{\mathrm{iso}}$*. Given* σ *in* $\Delta^+_{\mathbf{s}_v,\Pi}$*, then* $\sigma \bigcap q^*(M_{\mathbb{R}})$ *is identified as a chamber* $\overline{\sigma}$ *in* $\mathfrak{D}_{\overline{\mathbf{s}}}$ *via* $q^* : \overline{M}_{\mathbb{R}} \hookrightarrow M_{\mathbb{R}}$*. Moreover,* $\{\overline{\sigma} \mid \sigma \in \Delta^+_{\mathbf{s}_v,\Pi}\}$ *form a complex in* $\mathfrak{D}_{\overline{\mathbf{s}}}$ *which we denote by* $\overline{\Delta}^+_{\mathbf{s}_v,\Pi}$*.*

By previous lemma, for each chamber in $\mathcal{C}^+_{v'}$ in $\Delta^+_{\mathbf{s}_v,\Pi}$, there is a corresponding chamber in $\mathfrak{D}_{\overline{\mathbf{s}}}$ which we denote by $\overline{\mathcal{C}^+_{v'}}$.

Proposition 3.3.4. *Given* σ, σ' *in* $\Delta^+_{\mathbf{s}_v,\Pi}$*, the following diagram commutes:*

$$\begin{tikzcd}
T_{\overline{N}^\circ \oplus \overline{M},\overline{\sigma}} \arrow[r, hook] \arrow[d, dashed, "\mathfrak{p}_{\overline{\sigma},\overline{\sigma'}}"] & T_{N\oplus M,\sigma} \arrow[d, dashed, "\mathfrak{p}_{\sigma,\sigma'}"] \\
T_{\overline{N}^\circ \oplus \overline{M},\overline{\sigma'}} \arrow[r, hook] & T_{N\oplus M,\sigma'}
\end{tikzcd}$$

Here, the horizontal rows are closed embeddings of tori induced by the canonical quotient homomorphism of lattices $\tilde{q} : M \oplus N \twoheadrightarrow \overline{M}^\circ \oplus \overline{N}$*.*

Proof. It suffices to prove the diagram on the level of function fields:

$$\begin{tikzcd}
\mathbb{C}(M \oplus N) \arrow[r, two heads, "\tilde{q}"] \arrow[d, dashed, "\mathfrak{p}_{\overline{\gamma}}"] & \mathbb{C}(\overline{M}^\circ \oplus \overline{N}) \arrow[d, dashed, "\mathfrak{p}_{\gamma}"] \\
\mathbb{C}(M \oplus N) \arrow[r, two heads, "\tilde{q}"] & \mathbb{C}\left(\overline{M}^\circ \oplus \overline{N}\right)
\end{tikzcd}$$

Here, γ(resp. $\overline{\gamma}$) is a path in $\mathfrak{D}_{\mathbf{s}}$ (resp. $\mathfrak{D}_{\overline{\mathbf{s}}}$) connecting chambers σ and σ' (resp. $\overline{\sigma}$ and $\overline{\sigma'}$). By Proposition 2.3.5, $\mathfrak{p}_\gamma$ is contained in G^{Π} and $\mathfrak{p}_{\overline{\gamma}} = \overline{\mathfrak{p}_\gamma}$. Thus, the commutativity the diagram follows from Lemma 2.3.4. □

Similar to the case of $\Delta^+_{\mathbf{s}_v,\Pi}$, we can define

$$\mathcal{A}_{\mathrm{prin},\overline{\Delta}^+_{\mathbf{s}_v,\Pi}} = \bigcup_{\overline{\sigma} \in \overline{\Delta}^+_{\mathbf{s}_v,\Pi}} T_{\overline{N}^\circ \oplus \overline{M},\overline{\sigma}}$$

By Proposition 3.3.4, we have the following corollary:

COROLLARY 3.3.5. *$\mathcal{A}_{\mathrm{prin},\overline{\Delta}^+_{\mathbf{s}_v,\Pi}}$ is a closed subvariety of $\mathcal{A}_{\mathrm{prin},\Delta^+_{\mathbf{s}_v,\Pi}}$.*

PROPOSITION 3.3.6. *For each $\mathbf{s}_v$ in $[\mathbf{s}]^{\mathrm{iso}}$, we have*

$$\mathcal{A}_{\mathrm{prin},\overline{\Delta}^+_{\mathbf{s}_v,\Pi}} \simeq \mathcal{A}_{\mathrm{prin},\overline{\mathbf{s}}}$$

PROOF. Given $\mathbf{s}_v, \mathbf{s}_{v'}$ in $[\mathbf{s}]^{\mathrm{iso}}$, we have

$$\mathcal{A}_{\mathrm{prin},\Delta^+_{\mathbf{s}_v,\Pi}} \simeq \mathcal{A}_{\mathrm{prin},\Delta^+_{\mathbf{s}_{v'},\Pi}}$$

Moreover, under the above isomorphism restricts to an isomorphism between subvarieties:

$$\mathcal{A}_{\mathrm{prin},\overline{\Delta}^+_{\mathbf{s}_v,\Pi}} \simeq \mathcal{A}_{\mathrm{prin},\overline{\Delta}^+_{\mathbf{s}_{v'},\Pi}}$$

Thus, to prove the proposition, it suffices to prove for the case where $\mathbf{s}_v = \mathbf{s}$, which follows from section 4 of [**GHKK**]. $\square$

Define

$$\overline{\Delta} = \bigcup_{\mathbf{s}_v \in [\mathbf{s}]^{\mathrm{iso}}} \overline{\Delta}^+_{\mathbf{s}_v,\Pi}$$

and

$$\mathcal{A}_{\mathrm{prin},\overline{\Delta}} = \bigcup_{\mathbf{s}_v \in [\mathbf{s}]^{\mathrm{iso}}} \mathcal{A}_{\mathrm{prin},\overline{\Delta}^+_{\mathbf{s}_v,\Pi}}$$

Moreover, we have the following theorem:

THEOREM 3.3.7. *Given $\mathbf{s}_v, \mathbf{s}_{v'}$ in $[\mathbf{s}]^{\mathrm{iso}}$, $\overline{\Delta}^+_{\mathbf{s}_v,\Pi}$ and $\overline{\Delta}^+_{\mathbf{s}_{v'},\Pi}$ either coincide or have empty intersection. In the latter case, $\overline{\Delta}^+_{\mathbf{s}_v,\Pi}$ and $\overline{\Delta}^+_{\mathbf{s}_{v'},\Pi}$ parametrize two non-equivalent cluster structures in the weak sense for the variety $\mathcal{A}_{\mathrm{prin},\overline{\Delta}}$. Moreover, each of these cluster structures is isomorphic to that of $\mathcal{A}_{\mathrm{prin},\overline{\mathbf{s}}}$.*

3.3.1. Folding and the admissible group of Looijenga pair. Given a Looijenga pair (Y, D) with toric model $p : (Y, D) \to (\overline{Y}, \overline{D})$, throughout the rest of this section, we assume that **the number of irreducible components of D is less or equal to** 5. By Proposition I.4.7 of [**L81**], $D^{\perp}$ has a natural basis of roots and the admissible group of (Y, D) is equal to the Weyl group $W(Y, D)$ generated by reflections with respect to these root basis elements. Let $\overline{\Sigma}$ be the fan in a lattice $\Lambda \simeq \mathbb{Z}^2$. Let further assume that (Y, D) is generic, that is, there is no infinitely near blow up for π. Let $\{E_i\}_{i=1}^{n}$ be the exceptional configuration corresponding to the given toric model $p : (Y, D) \to (\overline{Y}, \overline{D})$. Set $N = \mathbb{Z}^n$ with basis $\{e_i\}_{i\in I}$ where $I = \{1, \cdots, n\}$. For each $i \in I$, denote by $\overline{D}_i$ the center of the non toric blow up with exceptional locus E_i. Each $\overline{D}_i$ is an irreducible component of the boundary divisor $\overline{D}$ of the smooth toric surface $\overline{Y}$ and corresponds to a ray in $\overline{\Sigma}$ with the primitive generator w_i in Λ. If we assume that $w_1, \cdots, w_n$ generate the lattice Λ, then the map

$$\varphi : N \to \Lambda$$

$$e_i \mapsto w_i$$

is surjective. Moreover, if we fix an isomorphism $\wedge^2\Lambda \simeq \mathbb{Z}$, then wedge product on Λ pull-back to an integral valued skew-symmetric form $\{,\}$ on N:

$$\{n_1, n_2\} = \varphi(n_1) \wedge \varphi(n_2)$$

Thus, $(N, \{,\}, (e_i)_{i\in I})$ give us a seed data and we denote by $\mathbf{s}$ the seed $(e_i)_{i\in i}$. Observe that the kernel K of the skew-symmetric form on N has codimension 2. By section 5 of [**GHK15**], the corresponding $\mathcal{X}$ cluster variety fibers over T_{K^*} with generic fiber agreeing up to codimension 2 with a log Calabi-Yau surface deformation equivalent to $Y \backslash D$. Notice that folding construction naturally applies in this situation. Given $i, j \in I$, if $\{e_i, \} = \{e_j, \}$, then the exceptional curves E_i and E_j have the same center $\overline{D}_i = \overline{D}_j$ and we can naturally fold e_i and e_j together. More generally, let Π be a subgroup of the symmetric group S_n generated by involutions such that for any π in Π,

$$\{e_i, \cdot\} = \{e_{\pi \cdot i}, \cdot\}$$

Then the action of Π satisfies the folding condition and gives us a folded seed $\bar{\mathbf{s}}$. Moreover, we can identify Π as a subgroup of the Weyl group (Y, D). Indeed, given $i \in I$ and $j \in \Pi i \backslash \{i\}$, under the identification $D^{\perp} \simeq K$ given in Theorem 5.5 in[**GHK15**] , $e_i - e_j$ corresponds the root $E_i - E_j$. Let π be the involution (ij) in Π. Then the induced action of π on $D^{\perp}$ agrees with the reflection $r_{E_i - E_j}$ with respect to the root $E_i - E_j$ in $W(Y, D)$.

3.3.1.1. *The action of Admissible groups on scattering diagrams.* Now given an initial seed $\mathbf{s}$ constructed from a Looijenga pair (Y, D) with toric model $(\overline{Y}, \overline{D})$, let v be a vertex in $\mathfrak{T}_{\mathbf{s}}$ with root v_0. By section 5 of [**GHK15**], the birational map

$$\mu_{v_0,v} : T_{N,\mathbf{s}} \dashrightarrow T_{N,\mathbf{s}_v}$$

extends to a birational map between blowups of toric varieties,

$$\mu_{v_0,v} : \mathcal{Y}_{\mathbf{s}} \dashrightarrow \mathcal{Y}_{\mathbf{s}_v}$$

such that for $\phi \in T_{K^*}$ general, $\mu_{v_0,v}$ restrict to an biregular isomorphism $\mathcal{Y}_{\mathbf{s},\phi} \backslash \mathcal{D}_{\mathbf{s},\phi} \to \mathcal{Y}_{\mathbf{s}_v,\phi} \backslash \mathcal{D}_{\mathbf{s}_v,\phi}$. If $\mathbf{s}_v$ is an seed isomorphic to $\mathbf{s}$, then $(\mathcal{Y}_{\mathbf{s},\phi}, \mathcal{D}_{\mathbf{s},\phi})$ and $(\mathcal{Y}_{\mathbf{s}_v,\phi}, \mathcal{D}_{\mathbf{s}_v,\phi})$ have the same toric model $(\overline{Y}, \overline{D})$. Let $\{E_{i,\phi}\}_{i \in I}$ be the exceptional configuration of $\pi_1 : \mathcal{Y}_{\mathbf{s},\phi} \to (\overline{Y}, \overline{D})$. Suppose this biregular isomorphism extends to a biregular isomorphism$\mathcal{Y}_{\mathbf{s},\phi} \to \mathcal{Y}_{\mathbf{s}_v,\phi}$, then the birational map

$$\mu_{v_0,v} : (\overline{Y}, \overline{D}) \dashrightarrow (\overline{Y}, \overline{D})$$

factors as

$$\begin{tikzcd}[column sep=small]
 & (\mathcal{Y}_{\mathbf{s},\phi}, \mathcal{D}_{\mathbf{s},\phi}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
(\overline{Y}, \overline{D}) \arrow[rr, dashed, "\mu_{v_0,v}"] & & (\overline{Y}, \overline{D})
\end{tikzcd}$$

where π_2 is the blowing down of another exceptional configuration $\{E'_{i,\phi}\}_{i\in I}$ of π_1. Moreover, since π_2 fixes the boundary component, there exists an unique element γ in the admissible group $\mathrm{Adm}(Y, D)$ that maps $\{E_{i,\phi}\}_{i\in I}$ to $\{E'_{i,\phi}\}_{i\in I}$. Hence γ have an induced action on $\mathfrak{D}_{\mathbf{s}}$ via by the piecewise linear automorphism given by the cluster modular group.

3.3.2. Folding and degenerate loci of family of log Calabi-Yau surfaces. Let $\mathbf{s}$be seed obtained from a generic Looijenga pair (Y, D) with toric model $p : (Y, D) \to (\overline{Y}, \overline{D})$ and $\bar{\mathbf{s}}$ a seed folded from $\mathbf{s}$ via the action of a group $\mathbf{\Pi}$ on the index set I. Notice that folding procedure does not change the rank of the exchange matrix, so $\mathcal{X}$-cluster variety associated to $\bar{\mathbf{s}}$ is still a rank 2 cluster variety. Moreover, $\mathcal{X}_{\bar{\mathbf{s}}}$ as a subvariety of $\mathcal{X}_{\mathbf{s}}$ lies in the degenerate loci of $\mathcal{X}_{\mathbf{s}}$. Let $\overline{K}$ be the kernel of the skew-symmetric form on $\overline{N}$. Let $\overline{\mathcal{Y}}_{\bar{\mathbf{s}}}$ be the blow up of the toric variety given by $\bar{\mathbf{s}}$. Given a general $\overline{\phi}$ in $T_{\overline{K}^*}$, $\mathcal{X}_{\bar{\mathbf{s}},\phi}$ up to codimension 2 agrees with $\overline{\mathcal{Y}}_{\bar{\mathbf{s}},\overline{\phi}}$, which is blow up of of $\overline{Y}$ at a collection of distinct points $p_{\Pi 1}, \cdots p_{\Pi n}$ where $p_{\Pi i} \in \overline{D}_i$ taken with multiplicity $|\Pi i|$.

Example 3.3.8. Consider the cubic surface (Y, D) with toric model $p : (Y, D) \to (\mathbb{P}^2, \overline{D})$ where $\overline{D} = \overline{D}_1 + \overline{D}_2 + \overline{D}_3$ is a triangle of lines in $\mathbb{P}^2$. Let l be the class of the pull-back of a line in $\mathbb{P}^2$ via p. We have $D^\perp \simeq \mathbf{D}_4$ and admissible group of (Y, D) is the Weyl group $W(\mathbf{D}_4)$. Moreover, $e_1 - e_2, e_3 - e_4, e_5 - e_6, e_1 + e_2 + e_3$ form a basis of K which under the identification $K \simeq D^\perp$ are roots $E_1 - E_2, E_3 - E_4, E_5 - E_6, l - E_1 - E_2 - E_3$ form a simple root system $\mathbf{D}_4$ in $D^{\perp}$. Let Π be the group generated by involutions $(12), (34), (56)$. Then Π can be identified as the subgroup generated by reflections with respect to roots $E_1 - E_2$, $E_3 - E_4$, $E_5 - E_6$. Let $\mathbf{s}$ be the seed given by (Y, D) and $\overline{\mathbf{s}}$ be the folded seed given by the action of Π on the index set. Then given a very general $\overline{\phi} \in T_{\overline{K}^*}$, $\mathcal{X}_{\overline{\mathbf{s}},\overline{\phi}}$ up to codimension 2 agrees with a singular cubic surface having the following description. For we blow up at a interior point $p_{i,\overline{\phi}}$ in $\overline{D}_i$ twice for each i. Since we assume that $\overline{\phi}$ is general, $p_{1,\overline{\phi}}, p_{2,\overline{\phi}}, p_{3,\overline{\phi}}$ are not collinear and therefore we could blow down the three -2 curves simultaneously and get a singular cubic surface with three ordinary double points.

3.4. The action of Adm groups on scattering diagrams: positive cases

Fix an initial seed $\mathbf{s}_0$ of rank 2. Let U be a generic fiber of the family $\mathcal{X} \to T_{K^*}$ where K is the kernel of the skew-symmetric pairing $\{\cdot,\cdot\}$ on N and $\mathcal{X}$ the rank 2 cluster variety associated to $\mathbf{s}_0$. Given an edge $v \xrightarrow{k} v'$ in $\mathfrak{T}_{\mathbf{s}_0}$, the natural identifications of $\mathcal{X}^{\mathrm{trop}}(\mathbb{R})$ with $M_{\mathbf{s}_v,\mathbb{R}}$ and $M_{\mathbf{s}_{v'},\mathbb{R}}$ induce an piecewise linear isomorphism $\mu^T_{v,v'} : M_{\mathbf{s}_v,\mathbb{R}} \to M_{\mathbf{s}_{v'},\mathbb{R}}$ as showed in the following diagram:

$$\begin{tikzcd} M_{\mathbf{s}_v,\mathbb{R}} & \mathcal{X}^{\mathrm{trop}}(\mathbb{R}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & M_{\mathbf{s}_{v'},\mathbb{R}} \arrow[from=1-1, to=1-3, bend left, "\mu^T_{v,v'}"] \end{tikzcd}$$

For each vertex v in $\mathfrak{T}_{\mathbf{s}_0}$, the map

$$p^* : N \longrightarrow M$$
$$n \mapsto \{n, \cdot\}$$

identifies $U^{\text{trop}}(\mathbb{Z})$ as a sublattice $U_{\mathbf{s}_v}$ of $M_{\mathbf{s}_v}$ where $U_{\mathbf{s}_v}$ is the image of the p^* map in $M_{\mathbf{s}_v}$. Moreover, since $\mu^T_{v,v'}$ restricts to an piecewise linear isomorphism $U_{\mathbf{s}_v,\mathbb{R}} \to U_{\mathbf{s}_{v'},\mathbb{R}}$, we obtain a canonical identification of $U^{\text{trop}}(\mathbb{Z})$ as a subset of $\mathcal{X}^{\text{trop}}(\mathbb{Z})$.

Let (Y, D) be a Looijenga pair such that $Y\backslash D = U$. Dualize the exact sequence

$$0 \to \langle D_i \rangle \to \text{Pic}(Y) \to \text{Pic}\,(U) \to 0$$

we obtain the exact sequence

$$0 \leftarrow \langle D_i \rangle^* \leftarrow \text{Pic}(Y) \leftarrow \text{Pic}\,(U)^* \leftarrow 0 \tag{3.4.1}$$

Comparing 3.4.1 with the exact sequence

$$0 \to D^{\perp} \to \text{Pic}(Y) \to \langle D_i \rangle^* \to 0$$

we can identify $(D^{\perp})^*$ with $\text{Pic}(U)$. Tropicalize the fibration $\mathcal{X} \to T_{K^*}$, we obtain the exact sequence

$$0 \to U^{\text{trop}}(\mathbb{Z}) \to \mathcal{X}^{\text{trop}}(\mathbb{Z}) \to K^* \simeq (D^{\perp})^* \simeq \text{Pic}(U) \to 0$$

THEOREM 3.4.1. *Fix a initial seed $\mathbf{s}_0$ for the root v_0. Given a vertex v in $\mathfrak{T}_{v_0}$, let $\{\mathbf{g}_i^v\}_{i=1}^n$ be the set of $\mathbf{g}$-vectors of $\mathcal{C}_v^+ \subset \mathfrak{D}_{\mathbf{s}_0}$. Let $\{E_i^{v_0}\}_{i=1}^n$ and $\{E_i^v\}_{i=1}^n$ be exceptional configurations for generic fibers in $\mathcal{Y}^{v_0} \to T_{K^*}$ and $\mathcal{Y}^v \to T_{K^*}$ respectively. Let $\mathring{E}_i^{v_0}$ and $\mathring{E}_i^{\,v}$ be classes we obtain after restricting to the complement of boundary divisors. Define $\mathring{E}_i^{v_0,v} = \sum_{j=1}^n \mathbf{g}_{ji}^v \mathring{E}_i^{v_0}$. Then for each $i \in I$,*

$$\mu^*_{v_0,v}\left(\mathring{E}_i^v\right) = \mathring{E}_i^{v_0,v}$$

PROOF. It suffices to prove for the case where v is adjacent to v_0 with an edge labelled by k. Then $\mathbf{g}_i^{v_0} = f_i$ for all i. Choose a generic period ϕ in T_{K^*} so that both $\mathcal{Y}_\phi^v$ and $\mathcal{Y}_\phi^{v_0}$ are blowup of toric varieties $\overline{Y}^{v_0}$ and $\overline{Y}^v$ at distinct points in the smooth locus of $\overline{D}^{v_0}$ and $\overline{D}^{v}$. Let $\overline{\Sigma}^{v_0}$ and $\overline{\Sigma}^{v}$ be fans for $\overline{Y}^{v_0}, \overline{Y}^{v}$ respectively. We could assume that ν_k is contained in both $\overline{\Sigma}^{v_0}$ and $\overline{\Sigma}^{v}$. Then the lattice map $\overline{N} \to \overline{N}/\mathbb{Z} \cdot \nu_k$ defines generic $\mathbb{P}^1$- fibrations $\overline{Y}^{v_0} \to \mathbb{P}^1$ and $\overline{Y}^{v} \to \mathbb{P}^1$. Composed with the blowup morphisms $\varphi^{v_0} : \mathcal{Y}^{v_0}_{\phi} \to \overline{Y}^{v_0}$ and $\varphi^{v} : \mathcal{Y}^{v}_{\phi} \to \overline{Y}^{v}$, we obtain generic $\mathbb{P}^1$-fibrations $\pi^{v_0} : \mathcal{Y}^{v_0}_{\phi} \to \mathbb{P}^1$ and $\pi^{v} : \mathcal{Y}^{v}_{\phi} \to \mathbb{P}^1$.

The mutation map $\mu_k : T_{N,\mathbf{s}_0} \dashrightarrow T_{N,\mathbf{s}_v}$ extends to a birational map $\mu_k : \mathcal{Y}^{v_0} \dashrightarrow \mathcal{Y}^{v}$ that is regular outside codimension 2 loci. Moreover, μ_k restricts to a biregular isomorphism between $\mathcal{Y}^{v_0}_{\phi}$ and $\mathcal{Y}^{v}_{\phi}$ that is analogous to elementary transformations in classical algebraic geometry. Under this isomorphism,

$$\mu_k^* \left(E_i^v\right) = \begin{cases} F_k^{v_0} - E_k^{v_0} & i = k \\ E_i^{v_0} & \text{otherwise} \end{cases}$$

where $F_k^{v_0}$ is the class of a fiber π^{v_0}. Moreover, we have

$$\begin{aligned} F_k^{v_0} &= \sum_{i=1}^{n} [\epsilon_{ki}^{v_0}]_+ \left(E_i^{v_0} + D_{-v_i}^{v_0}\right) \\ &= \sum_{i=1}^{n} [-\epsilon_{ki}^{v_0}]_+ \left(E_i^{v_0} + D_{-v_i}^{v_0}\right) \end{aligned}$$

Restricting to the complement of D^{v_0} and D^{v}, we have

$$\mathring{F}_k^{v_0} = \sum_{i=1}^{n} [\epsilon_{ki}^{v_0}]_+ \mathring{E}_i^{v_0} = \sum_{i=1}^{n} [-\epsilon_{ki}^{v_0}]_+ \mathring{E}_i^{v_0}$$

Since

$$\mathbf{g}_k^v = -\mathbf{g}_k^{v_0} + \sum_{i=1}^{n}[\epsilon_{ki}^{v_0}]_+\mathbf{g}_i^{v_0} - \left(\epsilon_k^{v_0}\right)^T$$
$$= -\mathbf{g}_k^{v_0} + \sum_{i=1}^{n}[-\epsilon_{ki}^{v_0}]_+\mathbf{g}_i^{v_0}$$

we have

$$E_i^{v_0,v} = \sum_{i=1}^{n}\mathbf{g}_{ik}^{v}E_i^{v_0}$$
$$= -E_k^{v_0} + \sum_{i=1}^{n}[-\epsilon_{ki}^{v_0}]_+\mathring{E}_i^{v_0}$$
$$= \mathring{F}_k^{v_0} - \mathring{E}_k^{v_0}$$
$$= \mu_k^*\left(\mathring{E}_k^{v}\right)$$

For $i \neq k$, $\mathbf{g}_i^v = \mathbf{g}_i^{v_0} = f_i$. Hence for $i \neq k$, $\mathring{E}_i^{v,v_0} = \mathring{E}_i^{v_0} = \mu_k^*(\mathring{E}_i^v)$. Therefore, the theorem holds. □

3.4.0.1. *The case of A_3-quiver: Blowing up $\mathbb{P}^2$ at three points.* Consider

$$\epsilon = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

The fan consisting of three rays generated by $-v_i (i = 1, 2, 3)$ gives us $\mathbb{P}^2$. The element $e_1 + e_2 + e_3$ generates the kernel K. The mutation sequence $[1, 2, 1, 3, 1]$ gives us the Donaldson-Thomas transformation of the corresponding $\mathcal{X}$-cluster variety such that generic fiber U of $\mathcal{X} \to T_{K^*}$ is isomorphic to the blowup of $\mathbb{P}^2$ at 3 general points. Let $D = D_1 + D_2 + D_3$ be the triangle of lines connecting two of the three general points we blow up. Then after the identification of K with $D^\perp$, $e_1 + e_2 + e_3$ corresponds to the root $l - E_1 - E_2 - E_3$ where l is the pullback of a line in $\mathbb{P}^2$ via blowup and

E_is are the three exceptional divisors. Split N as $N/K \oplus K$ via the following section:

$$\begin{aligned} N/K &\hookrightarrow N \\ \overline{e_1} &\mapsto e_1 \\ \overline{e_3} &\mapsto e_3 \end{aligned}$$

The DT-transformation is obtained via the mutation sequence

$$[0, 1, 2, 0]$$

Then we write the DT-transformation as the birational map:

$$\begin{aligned} \mathrm{DT} : \mathbb{P}^2 \times \mathbb{C}^* &\dashrightarrow \mathbb{P}^2 \times \mathbb{C}^* \\ \left(\left[1 : z^{-e_1} : z^{e_3}\right], z^{e_1+e_2+e_3}\right) &\longmapsto \left(\left[1 : z^{e_1} \frac{f(z^{e_3}, z^{e_2})}{f(z^{e_2}, z^{e_1})} : z^{-e_3} \frac{f(z^{e_1}, z^{e_3})}{f(z^{e_2}, z^{e_1})}\right], z^{-e_1-e_2-e_3}\right) \end{aligned}$$

where $f(z_1, z_2) = 1 + z_1 + z_1 z_2$. Define $\alpha = z^{e_1+e_2+e_3}$ and homogenize our equation by setting $\frac{X_1}{X_0} = z^{-e_1}$ and $\frac{X_2}{X_0} = z^{e_3}$, we get

$$\begin{aligned} [(X_0 : X_1 : X_2), \alpha] \longmapsto & ([X_1 (X_2 + \alpha X_1 + \alpha X_0) : X_2 (X_0 + X_2 + \alpha X_1) \\ & : X_0 (X_1 + X_0 + X_2)], \alpha^{-1}) \end{aligned}$$

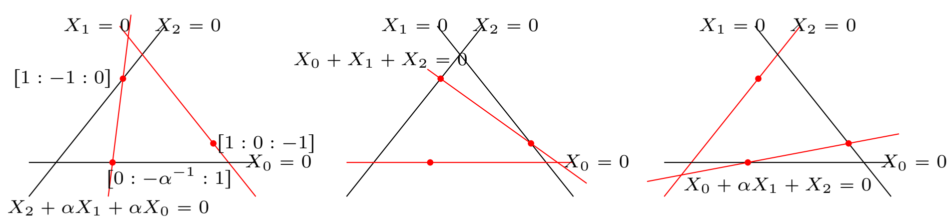
For $\alpha \neq 1$, the Donaldson-Thomas transformation restricts to a birational automorphism of $\mathbb{P}^2$:

$$\begin{aligned} \mathrm{DT}_\alpha : \mathbb{P}^2 \dashrightarrow & \mathbb{P}^2 \\ [X_0 : X_1 : X_2] \longmapsto & [X_1 (X_2 + \alpha X_1 + \alpha X_0) : X_2 (X_0 + X_2 + \alpha X_1) \\ & : X_0 (X_1 + X_0 + X_2)] \end{aligned}$$

We can minimally resolve DT_α as shown in the following diagram, where $\pi_1 : Y_\alpha \to \mathbb{P}^2$ is blowup of $\mathbb{P}^2$ at $p_1 = [1 : 0 : -1]$, $p_2 = [1 : -1 : 0]$ and $p_3 = [0 : -\alpha^{-1} : 1]$.

$$\begin{tikzcd} & Y_\alpha \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^2 \arrow[rr, dotted, "\mathrm{DT}_\alpha"] & & \mathbb{P}^2 \end{tikzcd}$$

To understand the morphism $\pi_2 : Y_\alpha \to \mathbb{P}^2$, we can look at the inverse image of each boundary divisor of $\mathbb{P}^2$ under DT_α, as shown in the following figure. Let $L_{ij}(1 \leqslant i < j \leqslant 3)$ be the line in $\mathbb{P}^2$ joining p_i and p_j. After blowing up at p_1, p_2, and p_3, the strict transform of L_{12}, L_{23}, L_{13} gives another exceptional configuration in Y_α and π_2 is the morphism that blows down this exceptional configuration.



Thus we see that the action of the DT-transformation as an element of the cluster modular group coincides with that of the reflection with respect to the root $l - E_1 - E_2 - E_3$.

Theorem 3.4.2. *Let (Y, D) be a generic Looijenga pair with $D^\perp \simeq \mathbf{D}_n(n \geqslant 4)$ or $\mathbf{E}_n(n = 6, 7, 8)$. Denote by $\mathcal{X}_{D^\perp}$ the corresponding $\mathcal{X}$-cluster variety and $\mathfrak{D}_{D^\perp}$ the scattering diagram living on $\mathcal{X}_{D^\perp}(\mathbb{R}^T)$. Then the admissible group of (Y, D) acts faithfully on the scattering diagram $\mathfrak{D}_{D^\perp}$ and preserves the cluster complex.*

Proof. By Proposition 3.2.1, we could assume that (Y, D) admits a toric model $(\mathbb{P}^2, \overline{D})$ where $\overline{D} = \overline{D}_1 + \overline{D}_2 + \overline{D}_3$ is the toric boundary divisor.

Thus $D^{\perp}$ has a root basis. Let $(E_i)_{i\in I}$ be the set of exceptional configurations and $(e_i)_{i\in I}$ the basis for the lattice $N \simeq \mathbb{Z}^n$ whose integral skew-symmetric form is given by the toric model $(\mathbb{P}^2, \overline{D})$. For simple roots of the form $E_{i_1} - E_{i_2}$ where E_{i_1} and E_{i_2} are two distinct exceptional curves along the same boundary component. Then $\{e_{i_1}, \cdot\} = \{e_{i_2}, \cdot\}$ and $E_{i_1} - E_{i_2}$ acts on the scattering diagram linearly by interchanging the dual basis elements $e^*_{i_1}$ and $e^*_{i_2}$. For simple roots of the form $l - E_{\epsilon_1} - E_{\epsilon_2} - E_{\epsilon_3}$ where l is the pull-back of the line and E_{ϵ_i} is an exceptional curve along $\overline{D}_i$. Then we have seen in the previous example that the reflection with respect these roots are also elements in the cluster modular group. Since the admissible group of (Y, D) is equal to $W(Y, D)$ which is generated by reflections with respect to simple roots, we conclude that the action of the admissible group preserves the cluster complex. $\square$

3.4.1. Factorizations of DT-transformations as cluster mutations for $D^{\perp} \simeq \mathbf{D}_n (n \geqslant 4)$.

LEMMA 3.4.3. *The following sequence of mutations ($n \geqslant 4$)*

$$[5,1,2,3,4,5],[6,1,2,3,4,6],[7,1,2,3,4,7]\cdots,[n+2,1,2,3,4,n+2] \tag{3.4.2}$$

gives the composition of reflections

$$\begin{aligned}&\left(r_{l-E_1-E_2-E_5} \circ r_{l-E_3-E_4-E_5}\right) \circ \left(r_{l-E_1-E_2-E_6} \circ r_{l-E_3-E_4-E_6}\right)\\ &\circ \cdots \circ \left(r_{l-E_1-E_2-E_{n+2}} \circ r_{l-E_3-E_4-E_{n+2}}\right)\end{aligned}$$

PROOF. This follows from the mutation formula for a single reflection of $r_{l-E_{\epsilon_1 - \epsilon_2 - \epsilon_3}}$ where $\epsilon_1 \in \{1, 2\}$, $\epsilon_2 \in \{3, 4\}$ and $\epsilon_3 \in \{5, 6, \cdots, n+1, n+2\}$. $\square$

THEOREM 3.4.4. *The Donaldson-Thomas transformation for $\mathcal{X}_{\mathbf{D}_n}$ is cluster.*

PROOF. First, we claim that after performing the sequence of reflections $w_5 \circ w_6 \circ \cdots \circ w_{n+2}$ where $w_i = r_{l-E_1-E_2-E_i} \circ r_{l-E_3-E_4-E_i}$, the $\mathbf{c}$-matrix, up to permutation by elements in Π, is of the form

$$\begin{pmatrix} C_n & \\ & -I_{n-2} \end{pmatrix}$$

where C_n is a 4 by 4 matrix and I_{n-2} is the identity matrix of rank $n-2$. We prove this claim by induction on n. The base case $n = 4$ is proved in the case of affine cubic surfaces. Suppose for $n = k(k \geqslant 4)$, the claim is true. When $n = k+1$, by induction hypothesis, after performing the sequence of reflections $w_5 \circ \cdots \circ w_{k+2}$, the $\mathbf{c}$-matrix is of the form

$$\begin{pmatrix} C_k & & * \\ & -I_{k-2} & * \\ & & * \end{pmatrix}$$

Here C_k is a 4 by 4 matrix and we do not know what the last column and row are yet. The last row has the last entry 1 and all other entries 0 and the last column have the first 4 entries and the last entry equal to 1 and all other entries 0. Indeed, by the mutation formula for the $\mathbf{c}$-vectors, since we never mutate in the $(k+3)$-th direction, the row has all but the last entries equal to 0. Denote the i^{th}-column of the $\mathbf{c}$-matrix by $\mathbf{c}_i$. To see that the last column is of the desired form, notice that by the action of the Weyl group on $D^\perp \simeq K$,

$$\mathbf{c}_{k+3} - \mathbf{c}_5 = e_1 + e_2 + e_3 + e_4 + e_5 + e_{k+3}$$

Since $\mathbf{c}_5 = -e_5$, we know that $\mathbf{c}_{k+3} = e_1 + e_2 + e_3 + e_4 + e_{k+3}$. Hence if we continue to perform the last reflection, the $\mathbf{c}$-matrix will be of the form

$$\begin{pmatrix} C_{k+1} & & * \\ & -I_{k-2} & * \\ & & * \end{pmatrix}$$

Again, by the action of the Weyl group, we know that the last column should be $-e_{k+3}$. Hence the $\mathbf{c}$-matrix is of the desired form for $n = k + 3$.

Denote by $\mathcal{C}^+_\omega$ the cluster chamber we obtain after performing the sequence of reflections $w_5 \circ w_6 \circ \cdots \circ w_{n+2}$. Now, to show that Donaldson-Thomas transformation for $\mathcal{X}_{\mathbf{D}_n}$ is cluster, it remains to show that the chamber $\mathcal{C}^+_\omega$ and $\mathcal{C}^-_{\mathbf{s}}$ are mutationally equivalent. This follows from the fact that the cluster complex of the scattering diagram of the Kronecker 2-quiver fill the whole space except for a single ray. $\square$

3.4.2. Application of folding to cases where $D^\perp \simeq \mathbf{D}_n$. Now let us apply the folding procedure to the case where $D^\perp \simeq \mathbf{D}_{2m} (m \geqslant 2)$. Let Π be the subgroup of the symmetric group S_{2m+2} generated by involutions:

$$(12), (34), (56), \cdots, (2m+1\, 2m+2)$$

Recall that α_l $(1 \leqslant l \leqslant m-1)$ denotes the root $2l - E_1 - E_2 - E_3 - E_4 - E_{2l+3} - E_{2l+4}$. Denote by μ_{α_l} the compositions of the sequence of mutations at gives reflection with respect to α_l, namely,

$$[2l+3, 1, 2, 3, 4, 2l+3,\ \ 2l+4, 1, 2, 3, 4, 2l+4]$$

Let $\mathcal{P}$ be the power set of $\{1, \cdots l\}$. Given $\varpi = \{i_1, \cdots, i_h\} \in \mathcal{P}$, let $\mathbf{s}_\varpi$ be the seed after performing the sequence of mutations $\mu_{\alpha_{i_h}} \circ \cdots \circ \mu_{\alpha_{i_1}}$. Since the Weyl group generated by r_{α_l} is abelian, $\mathcal{C}^+_{\mathbf{s}_\varpi}$ is independent of the choice of ordering of elements in ϖ. We have the following conjecture:

Conjecture 3.4.1. For any two distinct elements ϖ, ϖ' in $\mathcal{P}$, $\overline{\Delta}^+_{\mathbf{s}_\varpi,\Pi}$ and $\overline{\Delta}^+_{\mathbf{s}_{\varpi'},\Pi}$ has empty intersection.

The above conjecture is based on the numerical evidence that the chambers in $\Delta^+_{\mathbf{s},\Pi}$ is contained in the region in $\mathcal{X}^{\text{trop}}_{\mathbf{D}_{2m}}$ cut out by the equation

$$\left(z^{e_1+e_3+e_5}\right)^T \geqslant 0, \left(z^{e_1+e_3+e_7}\right)^T \geqslant 0, \cdots, \left(z^{e_1+e_3+e_{2m+1}}\right)^T \geqslant 0$$

while given any nonempty subset ϖ in $\mathcal{P}$, $\mathcal{C}^+_{\mathbf{s}_\varpi}$ is outside the above region. Moreover, we have the following corollary as a consequence of the previous conjecture:

Corollary 3.4.5. *$\mathcal{A}_{\text{prin},\overline{\Delta}}$ is a cluster variety with 2^{m-1} non-equivalent cluster structures, parametrized by elements in the subgroup of $W(\mathbf{D}_{2m})$ generated by reflections with respect roots $\alpha_l (1 \leqslant l \leqslant m-1)$.*

For right now, let us be less ambitious. Suppose $D^\perp \simeq \mathbf{D}_n$ for $n \geqslant 4$. Let Π' be the subgroup of the symmetric group S_{n+2} generated by involutions:

$$(12), (34), (56), (67), (78), \cdots, (n+1, n+2)$$

Then the folded seed $\bar{\mathbf{s}}$ given by the action of Π' has the following exchange matrix:

$$\begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ n-2 & n-2 & 0 \end{pmatrix}$$

Among the orbit of the Weyl group, the only chamber that is invariant under the action of Π' on the scattering diagram is given by the one after applying the sequence of reflections $w_5 \circ \cdots \circ w_{n+2}$ where for $5 \leqslant l \leqslant n+2$, w_l is the composition of reflections $r_{l-E_1-E_2-E_l} \circ r_{l-E_3-E_4-E_l}$. We denote the seed we obtain after performing this sequence of reflections by $\mathbf{s}_\omega$.

THEOREM 3.4.6. *The two subfans* $\overline{\Delta}^+_{\mathbf{s},\Pi'}$ *and* $\overline{\Delta}^+_{\mathbf{s}_\omega,\mathbf{\Pi}'}$ *have empty intersection in* $\mathfrak{D}_{\overline{\mathbf{s}}}$. *Moreover,* $\overline{\Delta}^+_{\mathbf{s},\Pi'}$ *coincide with* $\Delta^+_{\overline{\mathbf{s}}}$ *and* $\overline{\Delta}^+_{\mathbf{s}_\omega,\Pi'}$ *with* $\Delta^-_{\overline{\mathbf{s}}}$.

PROOF. Since $w_5 \circ \cdots \circ w_{n+2}$ is invariant under conjugation by elements in Π', we know that $\overline{\mathcal{C}}^+_\omega$ is a chamber in $\mathfrak{D}_{\overline{\mathbf{s}}}$. By the proof of Theorem 3.4.4, $\overline{\mathcal{C}}^+_\omega$ is mutationally equivalent to $\mathcal{C}^-_{\overline{\mathbf{s}}}$. Hence $\overline{\Delta}^+_{\mathbf{s}_\omega,\Pi'}$ with $\Delta^-_{\overline{\mathbf{s}}}$. By section 5.3 of [**TM**], the cluster modular group for $\mathcal{X}_{\overline{\mathbf{s}}}$ is isomorphic to $\mathbb{Z}$ and any element in it is given by an alternating sequence of mutations in the direction 0 and 1. Therefore we know that $\mathcal{C}^+_w$ is not contained in $\Delta^+_{\mathbf{s},\Pi'}$ and hence $\overline{\Delta}^+_{\mathbf{s},\Pi'}$ and $\overline{\Delta}^+_{\mathbf{s}_\omega,\mathbf{\Pi}'}$ has empty intersection in $\mathfrak{D}_{\overline{\mathbf{s}}}$. $\square$

Let $\tilde{\mathcal{A}}_{\text{prin},\overline{\mathbf{s}}}$ be the log Calabi-Yau variety we build using all chambers in $\Delta^+_{\overline{\mathbf{s}}} \bigcup \Delta^-_{\overline{\mathbf{s}}}$. By Theorem 3.3.7, we have the following corollary:

COROLLARY 3.4.7. *The variety* $\tilde{\mathcal{A}}_{\text{prin},\overline{\mathbf{s}}}$ *has two non-equivalent cluster structures in the weak sense, each isomorphic to that of* $\mathcal{A}_{\text{prin},\overline{\mathbf{s}}}$.

3.4.3. Factorizations of DT-transformations as cluster mutations for $D^\perp \simeq \mathbf{E}_n (n = 6, 7)$.

3.4.3.1. $D^\perp \simeq \mathbf{E}_6$. Given $(\mathbb{P}^2, \overline{D} = \overline{D}_1 + \overline{D}_2 + \overline{D}_3)$ where $\overline{D}$ is an anti-canonical cycle, blow up at two distinct interiors points on $\overline{D}_1$ with exceptional divisor E_1, E_2, blow up at three distinct interior points on $\overline{D}_2$ and $\overline{D}_3$ with exceptional divisors $E_3, E_4 E_5$ and E_6, E_7, E_8 respectively. Then the Looijenga pair we get (Y, D) will have $D^\perp \simeq \mathbf{E}_6$. To factor DT-transformations of $\mathcal{X}_{\mathbf{E}_6}$ as cluster transformations, we first perform the sequence of reflections

$$r_{l-E_1-E_3-E_6} \circ r_{l-E_1-E_4-E_7} \circ r_{l-E_1-E_5-E_8} \circ r_{l-E_2-E_3-E_6} \circ r_{l-E_2-E_4-E_7}$$
$$\circ\, r_{l-E_2-E_5-E_8} r_{l-E_1-E_3-E_6} \circ r_{l-E_1-E_4-E_7} \circ r_{l-E_1-E_5-E_8}$$

Then we append the mutation $[1, 2]$ and will get the DT-transformation of $\mathcal{X}_{\mathbf{E}_6}$. We can also simplify and factor $\mathrm{DT}_{\mathcal{X}_{\mathbf{E}_6}}$ as the following sequence of mutations:

$$[1, 3, 6, 4, 7, 5, 8, 1,\ 2, 3, 6, 4, 7, 5, 8, 2,\ 1, 3, 6, 4, 7, 5, 8, 2]$$

3.4.3.2. $D^{\perp} \simeq \mathbf{E}_7$. Given the Looijenga pair (Y, D) with $D^{\perp} \simeq \mathbf{E}_6$, if we blow up at another distinct interior point on D_3 with exceptional divisor E_9, then the new Looijenga pair $(Y^{'}, D^{'})$ will have $D^{'\perp} \simeq \mathbf{E}_7$. We could factor the DT-transformations of $\mathcal{X}_{\mathbf{E}_7}$ as the following sequence of reflections:

$$r_{l-E_3-E_1-E_6} \circ r_{l-E_3-E_2-E_7} \circ r_{l-E_4-E_1-E_6} \circ r_{l-E_4-E_2-E_7} \circ r_{l-E_5-E_1-E_6} \circ r_{l-E_5-E_2-E_7}$$

$$r_{l-E_3-E_1-E_8} \circ r_{l-E_3-E_2-E_9} \circ r_{l-E_4-E_1-E_8} \circ r_{l-E_4-E_2-E_9} \circ r_{l-E_5-E_1-E_8} \circ r_{l-E_5-E_2-E_9}$$

$$r_{l-E_3-E_1-E_6} \circ r_{l-E_3-E_2-E_7} \circ r_{l-E_4-E_1-E_6} \circ r_{l-E_4-E_2-E_7} \circ r_{l-E_5-E_1-E_6} \circ r_{l-E_5-E_2-E_7}$$

Equivalently, we can factor $\mathrm{DT}_{\mathcal{X}_{\mathbf{E}_7}}$ as the following sequence of mutations:

$$[3, 1, 2, 6, 7, 3,\ 4, 1, 2, 6, 7, 4,\ 5, 1, 2, 6, 7, 5,$$

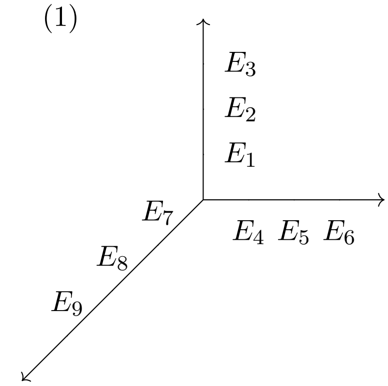
$$3, 6, 7, 8, 9, 3,\ 4, 6, 7, 8, 9, 4,\ 5, 6, 7, 8, 9, 5,$$

$$3, 8, 9, 1, 2, 3,\ 4, 8, 9, 1, 2, 4,\ 5, 8, 9, 1, 2, 5]$$

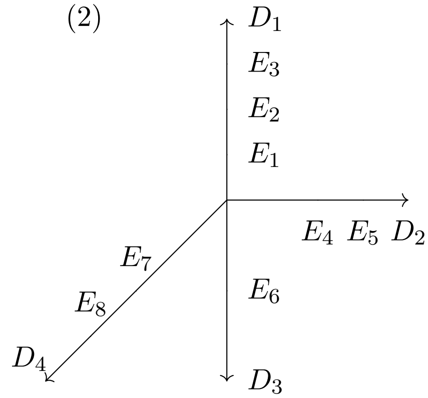
3.5. The negative semidefinite cases

3.5.1. Strictly negative semidefinite cases. In the following figure, we give the list of minimal strictly negative semidefinite Looijenga pair whose Mordell-Weil group is nontrivial. The toric model we choose for each pari are given by the fan in each illustration.

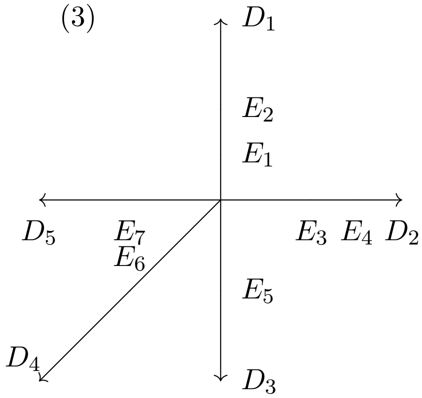
(1)


E_3
E_2
E_1
E_7
E_4 E_5 E_6
E_8
E_9

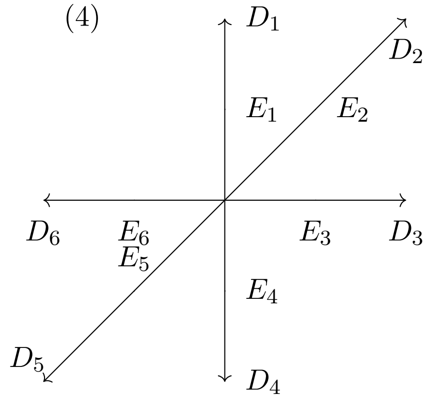
(2)


D_1
E_3
E_2
E_1
E_7
E_4 E_5 D_2
E_8
E_6
D_4
D_3

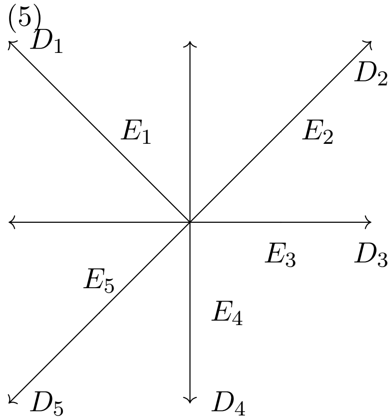
(3)


D_1
E_2
E_1
D_5 E_7 E_6
E_3 E_4 D_2
E_5
D_4
D_3

(4)


D_1
D_2
E_1
E_2
D_6 E_6 E_5
E_3 D_3
E_4
D_5
D_4

(5)


D_1
D_2
E_1
E_2
E_3 D_3
E_5
E_4
D_5
D_4

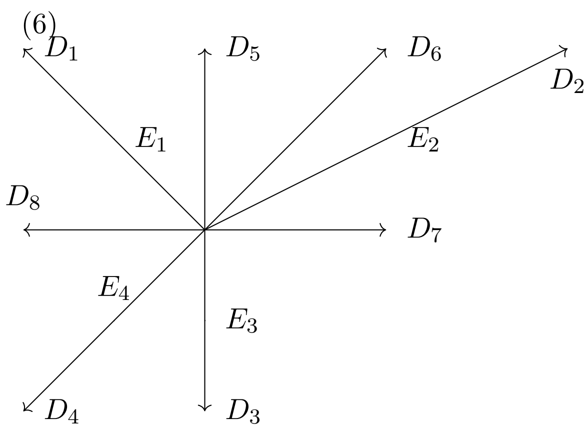


Figure 3.5.1. Minimal strictly negative semidefinite cases

Next, in each case, we give explicit descriptions of the subgroups of Mordell-Weil groups realized as subgroups in cluster modular groups. For the first three cases, the number of irreducible components of D is less or equal to 5. By Lemma 5.4 in [**GHK12**], in this three cases, the admissible groups are equal to Weyl groups. For case (1), up to conjugation by elements in Π, the Weyl group is generated by reflections with respect to roots of the form $L - E_1 - E_4 - E_7$ and $E_1 - E_2$. We know that both kinds of reflections are elements in the cluster modular group. Hence the whole Weyl group embeds into the cluster modular group.

For case (2), let $(\overline{Y}, \overline{D})$ be the Hirzebruch surface F_1 where $\overline{D_3}$ is the unique section with negative self intersection -1 and $\overline{D}_1$ has self intersection 1. Choose $\pi : (Y, D) \to (\overline{Y}, \overline{D})$ as our toric model. The following roots form a basis for $D^{\perp}$:

$$E_1 - E_2, E_2 - E_3, E_4 - E_5, E_7 - E_8, \pi^*(\overline{D_1}) - E_1 - E_4 - E_7, \pi^*(\overline{D_2}) - E_1 - E_6$$

The mutation sequence $[1,6]$ gives the reflection with respect to the root $\pi^*(\overline{D_2}) - E_1 - E_6$. We know that the reflections with respect to the rest of root basis elements are part of the cluster modular group.

For case (3), the following roots form a basis for $D^\perp$:

$$E_1-E_2, E_3-E_4, \pi^*(\overline{D}_1+\overline{D}_5)-E_1-E_3-E_6, \pi^*(\overline{D}_2)-E_1-E_5, \pi^*\left(\overline{D}_1\right)-E_3-E_7$$

Similar to case (2), the mutation sequence $[1,5]$ and $[3,7]$ give the reflection corresponding to the root $\pi^*(\overline{D}_2) - E_1 - E_5$ and $\pi^*\left(\overline{D}_1\right) - E_3 - E_7$ respectively. And we know that the reflections with respect to the rest of root basis elements are part of the cluster modular group.

For case (4), the number of irreducible component of D is 6, however, by Corollary 9.20 in [**F16**], the Weyl group is still equal to the admissible group. If we choose E_5 as the section, then the Mordell-Weil group can be identified as the following sublattice in $D^\perp$:

$$\Gamma = \mathbb{Z}_{[D_2+2E_2-E_1-E_3]} \bigoplus \mathbb{Z}_{[D_2+D_3+E_2+E_3-E_1-E_4]} \bigoplus \mathbb{Z}_{[D_1+D_2+E_1+E_2-E_3-E_6]}$$

Given an element γ in Γ , denote by g_γ is image in the admissible group. The Mordell-Weil group is an abelian subgroup of the admissible group and to see that it embeds into the cluster modular group, it suffices to show that each generator of the above sublattice is contained in the cluster modular group. Indeed, the mutation sequence

$$[1,4,2,5,3,6,2,5]$$

gives the element $g_{[D_2+D_3+E_2+E_3-E_1-E_4]}$, the mutation sequence

$$[3,6,2,5,1,4,2,5]$$

gives the element $g_{[D_1+D_2+E_1+E_2-E_3-E_6]}$ and the mutation sequence

Case (5) and case (6) are two exceptional cases where the Weyl group has infinite index inside the admissible group. For case (5), if we choose E_1 as the section, then the Mordell-Weil group can be identified as the following sublattice in $D^{\perp}$:

$$\Gamma = \mathbb{Z}_{[D_3+D_4+E_3+E_4-E_2-E_5]} \bigoplus \mathbb{Z}_{[D_3+2E_3-E_2-E_4]}$$

The Weyl group is isomorphic to the affine Weyl group of $\mathbf{A_1}$ and therefore has rank 1 while the Mordell-Weil group has rank 2. So we know that the Weyl group has infinite index in the admissible group. The mutation sequence

$$[2, 5, 1, 3, 4, 1]$$

gives the element $g_{[D_3+2E_3-E_2-E_4]}$

For case (6), $D^{\perp}$ does not contain any (-2) class and therefore the Weyl group is trivial. However, if we choose the Mordell-Weil group can be identified as the following sublattice in $D^{\perp}$:

$$\Gamma = \mathbb{Z}\gamma \text{ where } \gamma = [D_4+D_3+3D_7+5D_2+3D_6+D_5-D_1-3E_1+4E_2+E_4-2E_3]$$

Since the Mordell-Weil group has rank 1 and the Weyl group is trivial, the Weyl group obviously has infinite index inside the admissible group. The mutation sequence

$$[3, 1, 2, 4, 3, 1, 2, 4]$$

gives the element g_{γ}.

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